

Calculus, week 5, Fall 2026

Recall

We have considered the series approximation

$$\begin{aligned} f(x) &\approx \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} x^k \\ &= f(a) + \frac{f'(a)}{1!} x + \frac{f''(a)}{2!} x^2 + \frac{f^{(3)}(a)}{3!} x^3 + \dots \end{aligned}$$

Q1:

For what x , does $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} x^k$ converge?

Q2:

If $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} x^k$ converge, does it converge to $f(x)$?

Thm (Taylor's Thm, Thm 12.6.1)

Assume $f(x)$ has $n+1$ continuous derivatives on an open interval I , $a \in I$,

Then for $x \in I$, we have $f(x) = f(a) + R_n(x) = f(a) + \int_a^x f^{(n+1)}(t) dt$

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!} x + \frac{f''(a)}{2!} x^2 + \dots + \frac{f^{(n)}(a)}{n!} x^n \\ &\quad + R_n(x) \end{aligned}$$

where

$$R_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t) \cdot (x-t)^n dt$$

is called the remainder.

Step 0 $\frac{pf}{(n=0)}$ $f(x) \quad x \in I$. Since

$$\int_0^x f'(t) dt \stackrel{a}{=} f(x) - f(0)$$

we have

$$\underline{f(x) = f(0) + \int_0^x f'(t) dt} \quad (\text{case } n=0)$$

$u(t) \cdot v'(t)$

Step 1 $(n=1)$

$$\text{Let } u(t) = f'(t), \quad v(t) = t - x$$

$$\Rightarrow u'(t) = f''(t), \quad v'(t) = 1$$

$$\stackrel{b}{\Rightarrow} \int_0^x f'(t) dt = \left(f'(t) \cdot (t-x) \right) \Big|_{t=0}^x - \int_0^x f''(t) (t-x) dt$$

$$\begin{aligned} & \stackrel{=0}{\cancel{f'(x)}} \cdot \stackrel{=0}{\cancel{(x-x)}} + \underset{t=0}{f'(0) \cdot (0-x)} = f'(0) \cdot x \end{aligned}$$

$$= f'(0) \cdot x + \int_0^x f''(t) \cdot (x-t) dt$$

So

$$f(x) = f(0) + f'(0) \cdot x + \int_0^x \underbrace{f''(t)}_{u(t)} \cdot \underbrace{(x-t)}_{v'(t)} dt$$

case $n=1$.

Step 2 $(n=2)$

$$\text{Let } u(t) = f''(t), \quad v(t) = -\frac{(x-t)^2}{2}$$

$$u'(t) = f^{(3)}(t), \quad v(t) = (x-t)$$

$$\begin{aligned} \Rightarrow \int_0^x f'(t) \cdot (x-t) dt &= \left(f''(t) \cdot \left(-\frac{(x-t)^2}{2}\right) \right) \Big|_{t=0}^x \\ &= \cancel{f''(x) \cdot \left(-\frac{(x-x)^2}{2}\right)} - f''(0) \cdot \left(-\frac{(x-0)^2}{2}\right) - \int_0^x f^{(3)}(t) \cdot \left(-\frac{(x-t)^2}{2}\right) dt \\ &= f''(0) \cdot \frac{x^2}{2} + \boxed{\frac{1}{2} \int_0^x f^{(3)}(t) \cdot (x-t)^2 dt} \quad R_2(x) \end{aligned}$$

$$\text{So } f(x) = f(0) + f'(0)x + f''(0) \frac{x^2}{2} + R_2(x)$$

case $n=2$

⋮

Step n:

$$\frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x-t)^n dt$$

$$f(x) = f(0) + f'(0)x + \dots + \frac{f^{(n)}(0)}{n!} x^n + \boxed{R_n(x)} \quad \#$$

Want:

A simpler form of $R_n(x)$

Need:

Thm (Second Mean Value Thm, Thm 5.9.3)

If u and v are continuous on $[a, b]$

and $v \geq 0$ on $[a, b]$, then there exists $c \in [a, b]$ such that

$$\int_a^b u(t) v(t) dt = u(c) \cdot \int_a^b v(t) dt$$

Such $u(c)$ is called the v -weight average of u on $[a, b]$ 加權平均

Example

Let $u(t) = t^2$, $v(t) = t$ on $[a, b] = [0, 2]$
poly $\Rightarrow$

① u, v are continuous on $[0, 2]$

$$v(t) = t \geq 0 \quad \text{on } [0, 2]$$

$\Rightarrow$ The hypotheses are satisfied.

② Calculation of integrals:

$$\begin{aligned} \bullet \int_0^2 u(t) v(t) dt &= \int_0^2 t^2 \cdot t dt = \left. \frac{t^4}{4} \right|_0^2 \\ &= 4 \end{aligned}$$

$$\bullet \int_0^2 v(t) dt = \int_0^2 t dt = \left. \frac{t^2}{2} \right|_0^2 = 2$$

③ Find c :

Thm: there exists $c \in [0, 2]$ s.t.

$$\int_0^2 u(t) v(t) dt = u(c) \int_0^2 v(t) dt.$$

By ②,

$$4 = \overset{C^2}{u(c)} \cdot 2$$

In fact, we can solve c directly:

$$c = \pm \sqrt{\frac{4}{2}}$$

Since $-\sqrt{2} \notin [0, 2]$, we can only choose
 $c = \sqrt{2} \in [0, 2]$

which makes

$$u(c) = 2 = \frac{\int_0^2 u(t) v(t) dt}{\int_0^2 v(t) dt.}$$

#

Example

Let $u(t) = \cos t$, $v(t) = e^t$, $[a, b] = [0, \frac{\pi}{2}]$

① About hypotheses:

$u(t) = \cos t$ and $v(t) = e^t$ are continuous
 $v(t) = e^t > 0$ on $[0, \frac{\pi}{2}] \Rightarrow \text{ok.}$

② Calculation of integrals:

$$\int_0^{\frac{\pi}{2}} u(t) v(t) dt = \int_0^{\frac{\pi}{2}} \cos t \cdot e^t dt$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \underbrace{\cos t}_{b(t)} \underbrace{e^t}_{a'(t)} dt & \stackrel{\textcircled{b}}{=} \underbrace{\cos t \cdot e^t}_{b(t) \cdot a(t)} \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \underbrace{(+\sin t)}_{b'(t) = -\cos t} \underbrace{e^t}_{a(t) = e^t} dt \\ & \stackrel{\textcircled{b}}{=} \cos t \cdot e^t \Big|_0^{\frac{\pi}{2}} + \left(\sin t \cdot e^t \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos t \cdot e^t dt \right) \end{aligned}$$

$$\Rightarrow 2. \int_0^{\frac{\pi}{2}} \cos t \cdot e^t dt = (\cos t \cdot e^t + \sin t \cdot e^t) \Big|_0^{\frac{\pi}{2}}$$

$$= 0 \cdot e^{\frac{\pi}{2}} + 1 \cdot e^{\frac{\pi}{2}} - (1 \cdot e^0 + 0 \cdot e^0)$$

$$= e^{\frac{\pi}{2}} - 1$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos t \cdot e^t dt = \frac{e^{\frac{\pi}{2}} - 1}{2}$$

$$\int_0^{\frac{\pi}{2}} u(t) v(t) dt = \frac{e^{\frac{\pi}{2}} - 1}{2}$$

$$\bullet \int_0^{\frac{\pi}{2}} v(t) dt = \int_0^{\frac{\pi}{2}} e^t dt = e^t \Big|_0^{\frac{\pi}{2}}$$

$$= e^{\frac{\pi}{2}} - 1$$

③ Find c :

Thm : There exists $c \in [0, \frac{\pi}{2}]$ s.t.

$$\cos c = u(c) = \frac{\int_0^{\frac{\pi}{2}} u(t) v(t) dt}{\int_0^{\frac{\pi}{2}} v(t) dt}$$

$$= \frac{\frac{e^{\frac{\pi}{2}} - 1}{2}}{\frac{e^{\frac{\pi}{2}} - 1}{1}} = \frac{1}{2}$$

In fact, we solve c explicitly :

$$c = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \in [0, \frac{\pi}{2}]$$

The v -weight average of u on $[0, \frac{\pi}{2}]$ is

$$u\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \quad \#$$

Back to the remainder:

$$R_n(x) = \frac{1}{n!} \int_0^x \underbrace{f^{(n+1)}(t)}_{u(t)} \cdot \underbrace{(x-t)^n}_{v(t)} dt$$

Apply 2nd Mean Value Thm:

Case 1: $x \geq 0$

Choose $u(t) = \underbrace{f^{(n+1)}(t)}_{\text{continuous}}$. $v(t) = \frac{(x-t)^n}{n!} \geq 0$

$t \in [0, x]$

Then there exists $c \in [0, x]$ s.t.

$$\begin{aligned} R_n(x) &= \int_0^x u(t) v(t) dt = u(c) \cdot \int_0^x v(t) dt \\ &= f^{(n+1)}(c) \cdot \int_0^x \frac{(x-t)^n}{n!} dt = \left(-\frac{(x-t)^{n+1}}{(n+1)!} \right)' \\ &= - \frac{(x-t)^{n+1}}{(n+1)!} \Big|_{t=0}^x \\ &= - \frac{(x-x)^{n+1}}{(n+1)!} + \frac{(x-0)^{n+1}}{(n+1)!} = \frac{x^{n+1}}{(n+1)!} \end{aligned}$$

$\Rightarrow$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

where $c \in [0, x]$
depends on x and f

Case 2: $x < 0$

Choose $u(t) = (-1)^n f^{(n+1)}(t)$
 $v(t) = \frac{(t-x)^n}{n!}$

By similar computation, we have

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

Conclusion:

Thm

The remainder can be written as

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

with some c between 0 and x .

In other words,

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

$c \neq 0$ in general

for some c between 0 and x .

Example

Consider

$$f(x) = e^x, \quad x \in (-\infty, \infty).$$

By Thm,

$$e^x = f(0) + f'(0) x + \frac{f^{(2)}(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some c between 0 and x .

Since

$$f'(x) = e^x, \quad f''(x) = e^x, \quad \dots$$

$$f^{(n)}(x) = e^x, \quad f^{(n+1)}(x) = e^x$$

we have

$$f(0) = f'(0) = \dots = f^{(n)}(0) = e^0 = 1$$

$$\Rightarrow e^x = \boxed{1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}} + \boxed{\frac{e^c}{(n+1)!} x^{n+1}}$$

for some c between 0 and x

Consider $n \rightarrow \infty$:

Note that

$$|R_n(x)| = \left| \frac{e^c}{(n+1)!} x^{n+1} \right| = e^{\textcircled{c}} \cdot \frac{|x|^{n+1}}{(n+1)!}$$

$f(x) = e^x$ is increasing
and $c \leq |x|$

$e^{|x|}$ is independent of n

Recall: $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0 \quad (a = |x|)$

$$\Rightarrow \lim_{n \rightarrow \infty} R_n(x) = 0 \quad \text{for any } x \in (-\infty, \infty)$$

So

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \forall x \in \mathbb{R}$$

Recall

$$\bullet \quad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

Taylor's Thm $+ \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$

where

$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x-t)^n dt$$
$$= \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some c between 0 and x

$$\bullet \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all } x \in \mathbb{R}$$

Example

$$\textcircled{1} \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

for all $x \in \mathbb{R}$

pf

By Taylor's Thm,

$$\sin x = \underbrace{\sin x|_{x=0}}_0 + \frac{1}{1!} \underbrace{(\sin x)'|_{x=0}}_1 \cdot x$$
$$+ \frac{1}{2!} \underbrace{(\sin x)''|_{x=0}}_0 \cdot x^2 + \dots +$$

$$+ \frac{1}{n!} \frac{d^n(\sin x)}{dx^n} \Big|_{x=0} \cdot x^n + R_n(x)$$

Since

$$(\sin x)' \Big|_{x=0} = \cos x \Big|_{x=0} = 1$$

$$\frac{d^2 \sin x}{dx^2} \Big|_{x=0} = (\cos x)' \Big|_{x=0} = -\sin x \Big|_{x=0} = 0$$

$$\frac{d^3 \sin x}{dx^3} \Big|_{x=0} = \frac{d(-\sin x)}{dx} \Big|_{x=0} = -\cos x \Big|_{x=0} = -1$$

$$\frac{d^4 \sin x}{dx^4} \Big|_{x=0} = \frac{d(-\cos x)}{dx} \Big|_{x=0} = \sin x \Big|_{x=0} = 0$$

$$\frac{d^5 \sin x}{dx^5} \Big|_{x=0} = \frac{d \sin x}{dx} \Big|_{x=0} = \cos x \Big|_{x=0} = 1$$

we have

$$\sin x = \frac{1}{1!} x + \frac{(-1)}{3!} x^3 + \frac{1}{5!} x^5 + \frac{(-1)}{7!} x^7 + \dots$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$+ R_n(x)$$

$\rightarrow 0$?

Q: $\lim_{n \rightarrow \infty} R_n(x) = ?$ $-1 \leq \cos c, -\sin c, -\cos c, \sin c \leq +1$

Since

$$0 \leq |R_n(x)| = \left| \frac{\frac{d^{n+1} \sin x}{dx^{n+1}} \Big|_{x=c}}{(n+1)!} \cdot x^{n+1} \right|$$

Recall:
 $\frac{a^n}{n!} \rightarrow 0$

$$\leq \frac{1}{(n+1)!} |x|^{n+1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

we have (by pinching thm 夾擠)

$$\lim_{n \rightarrow \infty} R_n(x) = 0 \quad \forall x \in \mathbb{R}.$$

Conclusion:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \forall x \in \mathbb{R}$$

② By a similar argument, we have

$$\begin{aligned} \cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}$$

$\forall x \in \mathbb{R}$

Remark

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

① In general, $\lim_{n \rightarrow \infty} R_n(x) \neq 0$. $\uparrow \uparrow$ If it's zero,
it's ^{usually} difficult to see $\lim_{n \rightarrow \infty} R_n(x) = 0$.

So we need other methods (the theory of power series) to obtain Taylor expansion of a function.

② Taylor's Thm can be applied to estimate the value of a function

Example

Let $f(x) = \cos x$

(a) Write the Taylor expansion of $f(x)$ at 0 up to x^2 .

sol

$$\begin{aligned} f(x) &= \underbrace{f(0)}_{\cos 0 = 1} + \frac{\underbrace{f'(0)}_{-\sin 0 = 0}}{1!} x + \frac{\underbrace{f''(0)}_{-\cos 0 = -1}}{2!} x^2 + R_2(x) \\ &= 1 - \frac{x^2}{2!} + R_2(x) \quad \text{(*)} \quad \# \end{aligned}$$

(b) Show that $|R_2(x)| \leq \frac{|x|^3}{6}$

Pf

By Taylor's Thm,

$$\begin{aligned} f(x) = \cos x &\Rightarrow f'(x) = -\sin x \\ f''(x) &= -\cos x \quad f'''(x) = \sin x \end{aligned}$$

$$R_2(x) = \frac{f'''(c)}{3!} x^3 \quad \text{for some } c$$

$$= \frac{\sin c}{6} x^3$$

$$\Rightarrow |R_2(x)| = \frac{|\sin c| \leq 1}{6} |x|^3 \leq \frac{|x|^3}{6} \quad \#$$

(c) Use this expansion to approximate $\cos(\frac{1}{5})$. Give an upper bound for the absolute error. $\left| \cos(\frac{1}{5}) - \text{your ans.} \right|$

Sol

⊗:

$$\cos\left(\frac{1}{5}\right) = 1 - \frac{1}{2!} \left(\frac{1}{5}\right)^2 + R_2\left(\frac{1}{5}\right) \\ \approx 1 - \frac{1}{2!} \left(\frac{1}{5}\right)^2 = 1 - \frac{1}{50} = \frac{49}{50}$$

And

$$\left| \cos\left(\frac{1}{5}\right) - \frac{49}{50} \right| = \left| R_2\left(\frac{1}{5}\right) \right|$$

$$\leq \frac{1}{6} \left| \frac{1}{5} \right|^3 = \frac{1}{750} \quad \#$$

Thm (Thm 12.7.1)

If f has $n+1$ continuous derivatives on an open interval I that contains a , then $\forall x \in I$,

$$f(x) = f(a) + \frac{f'(a)}{1!} \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 \\ + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t) (x-t)^n dt$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

for some c between a and x .

Example

Let $f(x) = \sqrt{x}$ (Note: $f'(x) = \frac{1}{\sqrt{x}} \cdot \frac{1}{2}$ which is NOT defined at $x=0$)

(a) Expand $f(x)$ at 4 up to $(x-4)^2$.

(b) Show for $4 \leq x \leq 5$,
 $0 \leq R_2(x) \leq \frac{(x-4)^3}{512}$

(c) Use this expansion to approximate $\sqrt{5}$.
 Give an upper bound for the absolute error.

Sol

$$(a) f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}}$$

Recall
 $(x^r)' = r \cdot x^{r-1}$

$$f''(x) = \frac{1}{2} \left(-\frac{1}{2}\right) \cdot x^{-\frac{1}{2}-1} = -\frac{1}{4} x^{-\frac{3}{2}}$$

$$\left(f'''(x) = -\frac{1}{4} \cdot \left(-\frac{3}{2}\right) \cdot x^{-\frac{3}{2}-1} = \frac{3}{8} x^{-\frac{5}{2}} \right)$$

$$\Rightarrow f(4) = \sqrt{4} = 2$$

$$f'(4) = \frac{1}{2} \frac{1}{\sqrt{4}} = \frac{1}{4}$$

$$f''(4) = -\frac{1}{4} \cdot \frac{1}{\sqrt{4^3}} = -\frac{1}{4} \cdot \frac{1}{2^3} = -\frac{1}{32}$$

$$\Rightarrow \sqrt{x} = f(x) = f(4) + \frac{f'(4)}{1!} (x-4) + \frac{f''(4)}{2!} (x-4)^2 + R_2(x)$$

$$= 2 + \frac{1}{4} (x-4) - \frac{1}{64} (x-4)^2 + R_2(x) \quad \#$$

(b)

$$R_2(x) = \frac{f'''(c)}{3!} (x-4)^3 \quad \text{for some } c \text{ between } 4 \text{ and } x$$

$$= \frac{1}{24} \cdot \frac{3}{8} c^{-\frac{5}{2}} \cdot (x-4)^3$$

$$= \frac{(x-4)^3}{16 c^{\frac{5}{2}}}$$

For $4 \leq x \leq 5$, we have $4 \leq c \leq 5$

$$\Rightarrow 32 = 2^5 = 4^{\frac{5}{2}} \leq c^{\frac{5}{2}}$$

$$\Rightarrow \frac{1}{16 c^{\frac{5}{2}}} \leq \frac{1}{32 \times 16} = \frac{1}{512}$$

$$\Rightarrow 0 \leq R_2(x) = \frac{(x-4)^3}{16 c^{\frac{5}{2}}} \leq \frac{(x-4)^3}{512} \quad \#$$

$\because 4 \leq x \leq 5$

(c) $\sqrt{5} = 2 + \frac{1}{4} (5-4) - \frac{1}{64} (5-4)^2 + R_2(5)$

$$\approx 2 + \frac{1}{4} - \frac{1}{64} = \frac{143}{64}$$

Furthermore,

$$0 \leq \sqrt{5} - \frac{143}{64} = R_2(5) \leq \frac{(5-4)^3}{512} = \frac{1}{512} \quad \#$$

Example

$$= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3$$

Let $f(x) = 4x^3 - 3x^2 + 5x - 1$

Compute Taylor expansion of $f(x)$ at 2

Sol

$$f'(x) = 12x^2 - 6x + 5 \Rightarrow f'(2) = 12 \cdot 4 - 6 \cdot 2 + 5 = 41$$

$$f''(x) = 24x - 6 \Rightarrow f''(2) = 48 - 6 = 42$$

$$\overset{\Rightarrow f^{(4)}(x)=0}{f''''(x) = 24} \Rightarrow f''''(2) = 24$$

$$\Rightarrow f(x) = \underbrace{f(2)}_{29} + \frac{f'(2)}{1!} (x-2) + \frac{f''(2)}{2!} (x-2)^2 + \frac{f'''(2)}{3!} (x-2)^3 + \underbrace{R_3(x)}_{=0} = 0 + \frac{1}{4!} f^{(4)}(c) (x-2)^4$$

$$= 29 + 41(x-2) + 21(x-2)^2 + 4(x-2)^3 \quad \#$$