

Calculus, week 4, Fall 2026

Thm (Thm 12.2.4, 12.2.5)

Let a_k, b_k, α, β be numbers

(i) $\sum_{k=0}^{\infty} a_k$ Converges to L

$\sum_{k=0}^{\infty} b_k$ Converges to M

$\Rightarrow \sum_{k=0}^{\infty} (\alpha a_k + \beta b_k)$ Converges to $\alpha L + \beta M$
 $\alpha=1, \beta=-1 \Rightarrow \sum (a_k - b_k)$
Conv.

(ii) If $\sum_{k=0}^{\infty} a_k$ Converges, then

$$\lim_{k \rightarrow \infty} a_k = 0$$

Equivalent, if $\lim_{k \rightarrow \infty} a_k \neq 0$, then
 $\sum_{k=0}^{\infty} a_k$ diverges.

Example

$$\textcircled{1} \sum_{k=0}^{\infty} \left(\left(\frac{1}{2} \right)^k + 2 \cdot \left(\frac{1}{3} \right)^k \right)$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^k + 2 \cdot \sum_{k=0}^{\infty} \left(\frac{1}{3} \right)^k$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad \text{if } |x| < 1$$

$$= \frac{1}{1 - \frac{1}{2}} + 2 \cdot \frac{1}{1 - \frac{1}{3}}$$

$$= \frac{1}{\frac{1}{2}} + 2 \cdot \frac{1}{\frac{2}{3}} = 2 + 3 = 5 \neq$$

② $\sum_{k=0}^{\infty} \frac{k}{k+1}$ diverges because

$$\lim_{k \rightarrow \infty} \frac{k/k}{(k+1)/k} = \lim_{k \rightarrow \infty} \frac{1}{1 + \frac{1}{k}} = 1 \neq 0 \neq$$

Remark

$$\lim_{k \rightarrow \infty} a_k \neq 0 \Rightarrow \sum_{k=0}^{\infty} a_k \text{ diverges}$$

BUT

$$\lim_{k \rightarrow \infty} a_k = 0 \not\Rightarrow \sum_{k=0}^{\infty} a_k \text{ converges}$$

NOT sure

Example

③ $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges and $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{k}} = 0$

Why it diverges? It diverges because

$$\begin{aligned}
 \sum_{k=1}^n \frac{1}{\sqrt{k}} &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} \quad \leftarrow \text{smallest} \\
 &\geq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{n}} + \dots + \frac{1}{\sqrt{n}} \\
 &= \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{n}} \rightarrow +\infty \text{ as } n \rightarrow \infty \\
 &\quad (\text{NOT bounded above})
 \end{aligned}$$

$\Rightarrow \left(\sum_{k=1}^n \frac{1}{\sqrt{k}} \right)_{n=1}^{\infty}$ is NOT bounded

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges. #

④ $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$ converges and $\lim_{k \rightarrow \infty} \left(\frac{1}{2}\right)^k = 0$ #

Convergence of series

Q: Does $\sum_{k=0}^{\infty} a_k$ converge?

We consider a basic and important

Case: $a_k \geq 0 \quad \forall k = 0, 1, 2, \dots$

Observe that if $a_k \geq 0 \quad \forall k$, the
 $\left(\sum_{k=0}^n a_k \right)_{n=1}^{\infty}$ is increasing:

$$a_0, \quad a_0 + a_1 \leq a_0 + a_1 + \underbrace{a_2}_{=0}, \quad \dots$$

Thm (Thm 12.3.1)

Assume $a_k \geq 0 \quad \forall k$. Then

$$\sum_{k=0}^{\infty} a_k \text{ converges}$$

$\Leftrightarrow \left(\sum_{k=0}^n a_k \right)_{n=0}^{\infty}$ is bounded above

That is, $\exists M$ s.t.

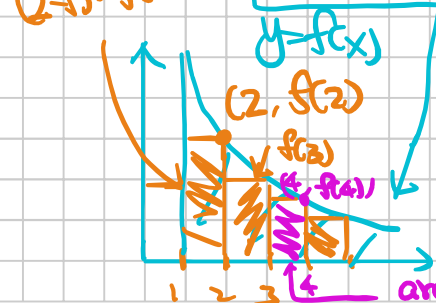
$$\sum_{k=0}^n a_k \leq M \quad \forall n$$

Thm (Integral Test, Thm 12.3.2)

If $f(x)$ is continuous, positive and decreasing on $[1, \infty)$, then

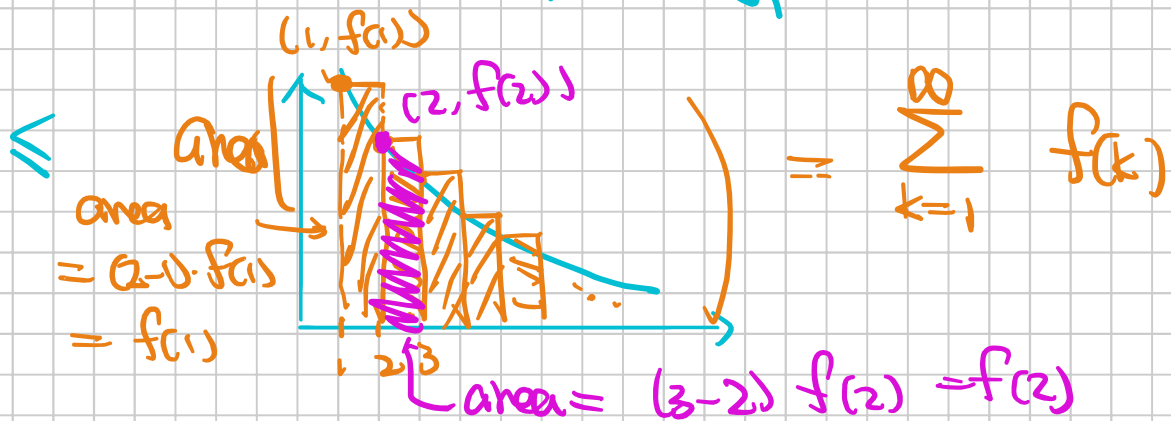
$$\sum_{k=1}^{\infty} \underbrace{f(k)}_{=a_k} \text{ Converges}$$

$$\Leftrightarrow \int_1^{\infty} f(x) dx \text{ Converges.}$$



$$\begin{aligned} \text{Area} &= f(2) + f(3) + f(4) + \dots \\ &= \sum_{k=2}^{\infty} f(k) \end{aligned}$$

$$\leq \text{Area} \left(\text{under } f(x) \right) = \int_1^{\infty} f(x) dx$$



Example

Consider

$$f(x) = \frac{1}{x^p} \quad (p > 0)$$

which is continuous, positive, decreasing on $[1, \infty)$

So, by Thm,

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Converges

$$\Leftrightarrow \int_1^{\infty} \frac{1}{x^p} dx \text{ Converges.}$$

Recall

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

$$\Leftrightarrow p > 1$$

So, in particular, we know:

- $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ ($p = \frac{1}{2} \leq 1$) diverges.

- $\sum_{k=1}^{\infty} \frac{1}{k}$ ($p = 1$) diverges

- $\sum_{k=1}^{\infty} \frac{1}{k^2}$ ($p = 2$) Converges

...

Thm (Basic Comparison Test, Thm 12.3.6) #

Let $a_k, b_k \geq 0$. Suppose
 $a_k \leq b_k$ for k large enough

(i) $\sum_{k=0}^{\infty} b_k$ converges $\Rightarrow \sum_{k=0}^{\infty} a_k$ Converges

(ii) $\sum_{k=0}^{\infty} a_k$ diverges
 $\Rightarrow \sum_{k=0}^{\infty} b_k$ diverges.

Example

① $\sum_{k=1}^{\infty} \frac{1}{2k^3+1}$ Converges because

$$\frac{1}{2k^3+1} < \frac{1}{k^3} \quad \forall k \geq 1$$

$k^3 + (k^3+1)$ Smaller

and $\sum_{k=1}^{\infty} \frac{1}{k^3}$ Converges ($p=3 > 1$). #

② $\sum_{k=1}^{\infty} \frac{k^3}{k^5+5k^4+7}$ Converges because

$$\frac{k^3}{k^5+5k^4+7} < \frac{k^3}{k^5} = \frac{1}{k^2} \quad \forall k \geq 1$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges #

③ $\sum_{k=1}^{\infty} \frac{1}{3k+1}$ diverges because

$$\frac{1}{3k+1} \geq \frac{1}{3k+k} = \frac{1}{4} \cdot \frac{1}{k} \quad \forall k \geq 1$$

and $\sum_{k=1}^{\infty} \frac{1}{4} \cdot \frac{1}{k} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k}$ diverges #

Thm (Limit Comparison Test, Thm 12.3.7)

Let $a_k, b_k > 0$. If

$$\lim_{k \rightarrow \infty} \frac{b_k}{a_k} \neq 0$$

then

$$\sum_{k=0}^{\infty} a_k \text{ Converges}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} b_k \text{ Converges}$$

Example

① $\sum_{k=1}^{\infty} \frac{1}{5^k - 3}$ Converges because

$$\lim_{k \rightarrow \infty} \frac{\frac{1}{5^k - 3}}{\frac{1}{5^k}} = \lim_{k \rightarrow \infty} \frac{1}{1 - \frac{3}{5^k}} = 1 \neq 0$$

and $\sum_{k=1}^{\infty} \frac{1}{5^k}$ Converges #

② $\sum_{k=1}^{\infty} \frac{3k^2 + 2k + 1}{k^3 + 1}$ diverges because

$\frac{3k^2 + 2k + 1}{k^3 + 1} \sim \frac{3k^2}{k^3} = \frac{3}{k}$

$$\lim_{k \rightarrow \infty} \frac{\frac{3k^2 + 2k + 1}{k^3 + 1}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{(3k^3 + 2k^2 + k)/k^3}{(k^3 + 1)/k^3}$$

$$= \lim_{k \rightarrow \infty} \frac{3 + 2\frac{1}{k} + \frac{1}{k^2}}{1 + \frac{1}{k^3}} = 3 \neq 0$$

and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. $\neq$

③ $\sum_{k=1}^{\infty} \frac{2k+5}{\sqrt{k^6+3k^3}}$ Converges because

$$\lim_{k \rightarrow \infty} \frac{\frac{2k+5}{\sqrt{k^6+3k^3}}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{(2k^3+5k^2)/k^3}{(\sqrt{k^6+3k^3})/k^3}$$

$$= \lim_{k \rightarrow \infty} \frac{2 + \frac{5}{k}}{\sqrt{\frac{k^6}{k^6} + 3\frac{k^3}{k^6}}} = 2 \neq 0$$

$\sqrt{k^6} = k^3$

And $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges $\neq$

Thm (Root Test, Thm 12.4.1)

Let $a_k \geq 0$. Suppose

$$\rho := \lim_{k \rightarrow \infty} \sqrt[k]{a_k}$$

$$\iff \sum_{k=0}^{\infty} x^k \quad \lim_{k \rightarrow \infty} \sqrt[k]{x^k} = x$$

(i) $\rho < 1 \Rightarrow \sum_{k=0}^{\infty} a_k$ Converges

(ii) $\rho > 1 \Rightarrow \sum_{k=0}^{\infty} a_k$ diverges

(iii) $\rho = 1 \Rightarrow$ NO conclusion

Example

① $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^k}$ Converge or not?

Sol

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{\frac{1}{(\ln k)^k}} = \lim_{k \rightarrow \infty} \frac{1}{\ln k} = 0 < 1$$

By Root Test, we conclude that

$\sum_{k=2}^{\infty} \frac{1}{(\ln k)^k}$ Converges. $\#$

② $\sum_{k=1}^{\infty} \frac{2^k}{k^3}$ Converge or not?

Sol

Recall: $\lim_{k \rightarrow \infty} \sqrt[k]{k} = 1$

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{\frac{2^k}{k^3}} = \lim_{k \rightarrow \infty} \frac{2}{(\sqrt[k]{k})^3}$$

$$= \frac{2}{1} = 2 > 1$$

$\Rightarrow \sum_{k=1}^{\infty} \frac{2^k}{k^3}$ diverges. #

Remark

If $\rho = \lim_{k \rightarrow \infty} \sqrt[k]{a_k} = 1$, then $\sum_{k=1}^{\infty} a_k$ may or may not converge.

Explicitly, consider: $\Rightarrow \lambda = \lim_{k \rightarrow \infty} \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \left(\frac{k}{k+1}\right)^2 = 1$

① $\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges and

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{1}{(\sqrt[k]{k})^2} \rightarrow 1$$

② $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges and $\Rightarrow \lambda = \lim_{k \rightarrow \infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{k}{k+1} = 1$

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{1}{\sqrt[k]{k}} \rightarrow 1 = 1$$

Thm (Ratio Test, Thm 12.4.2)

Let $a_k > 0$ and

$$\lambda = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k}$$

$$\sum_{k=0}^{\infty} x^k \quad \leftarrow \quad \lim_{k \rightarrow \infty} \frac{x^{k+1}}{x^k} = x$$

(i) $\lambda < 1 \Rightarrow \sum_{k=0}^{\infty} a_k$ Converges

(ii) $\lambda > 1 \Rightarrow \sum_{k=0}^{\infty} a_k$ diverges

(iii) $\lambda = 1 \Rightarrow$ NO conclusion.

Example

① $\sum_{k=0}^{\infty} \frac{1}{k!}$ $\leftarrow k! = 1 \cdot 2 \cdot 3 \dots k$

Converges because

Ratio Test $\lambda = \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \lim_{k \rightarrow \infty} \frac{1}{k+1} = 0 < 1$ #

② $\sum_{k=1}^{\infty} \frac{k}{10^k}$

$$\lambda = \lim_{k \rightarrow \infty} \frac{\frac{k+1}{10^{k+1}}}{\frac{k}{10^k}} = \lim_{k \rightarrow \infty} \frac{k+1}{k} \cdot \frac{1}{10}$$

Ratio Test $\Rightarrow$ $\sum_{k=1}^{\infty} \frac{k}{10^k} < 1$ Converges. #

③ $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

$\lambda = \lim_{k \rightarrow \infty} \frac{\frac{(k+1)^{k+1}}{(k+1)!}}{\frac{k^k}{k!}}$

Recall $\lim_{k \rightarrow \infty} \left(1 + \frac{x}{k}\right)^k = e^x$

$= \lim_{k \rightarrow \infty} \left(\frac{k+1}{k}\right)^k = \lim_{k \rightarrow \infty} \left(1 + \frac{1}{k}\right)^k$

$= e = 2.718 \dots > 1$

$\Rightarrow \sum_{k=1}^{\infty} \frac{k^k}{k!}$ diverges. #

Recall

Main question: Does $\sum_{k=0}^{\infty} a_k$ converge?

Step 1: (k-th) n-th term Test

1.1: $\lim_{k \rightarrow \infty} a_k \neq 0 \Rightarrow \sum_{k=0}^{\infty} a_k$ diverges

e.g. $\sum_{k=1}^{\infty} (1 - \frac{1}{k})$ diverges because

$$\lim_{k \rightarrow \infty} (1 - \frac{1}{k}) = 1 \neq 0$$

1-2: $\lim_{k \rightarrow \infty} a_k = 0 \Rightarrow$ Step 2

Step 2: Check $a_k \geq 0$?

2-1: $a_k \geq 0 \forall k \Rightarrow$ use the following tests

2-2: $a_k \leq 0 \Rightarrow$ Consider $-\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} (-a_k) \geq 0$
and use the following tests.

2-3: Some $a_k \geq 0$, some $a_k \leq 0$
 $\Rightarrow$ Step 3

Integral Test:

Assume $\exists f(x)$ which is continuous, positive, decreasing on $[1, \infty)$, and $a_k = f(k)$

Then

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} f(k) \quad \text{Converges}$$

$$\Leftrightarrow \int_1^{\infty} f(x) dx \quad \text{Converges.}$$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ conv.
 $\sum_{k=1}^{\infty} \frac{1}{k}$ div.

e.g. $\sum_{k=1}^{\infty} \frac{1}{k^p}$ Converges $\Leftrightarrow p > 1$

Basic Comparison Test

Assume $(0 \leq) a_k \leq b_k$

Then (i) $\sum_{k=0}^{\infty} b_k$ conv. $\Rightarrow \sum_{k=0}^{\infty} a_k$ conv.

(ii) $\sum_{k=0}^{\infty} a_k$ div. $\Rightarrow \sum_{k=0}^{\infty} b_k$ div.

Limit Comparison Test

Assume $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} \neq 0$ ($\neq \infty$) Then

$\sum_{k=0}^{\infty} a_k$ conv. $\Leftrightarrow \sum_{k=0}^{\infty} b_k$ conv.

eg. $\sum_{k=1}^{\infty} \frac{1}{k^2 + \sqrt{k}}$ conv. because $\lim_{k \rightarrow \infty} \frac{\frac{1}{k^2 + \sqrt{k}}}{\frac{1}{k^2}} = 1$
and $\sum \frac{1}{k^2}$ conv.

Root Test

Let $\lim_{k \rightarrow \infty} \sqrt[k]{a_k} = p$

Then (i) $p < 1 \Rightarrow \sum_{k=0}^{\infty} a_k$ conv.

(ii) $p > 1 \Rightarrow \sum_{k=0}^{\infty} a_k$ div.

Ratio Test

Let $\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lambda$. Then

(i) $\lambda < 1 \Rightarrow \sum_{k=0}^{\infty} a_k$ conv.

(ii) $\lambda > 1 \Rightarrow \sum_{k=0}^{\infty} a_k$ div.

Step 3 What if some $a_k > 0$, some $a_k < 0$?

eg. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

3-1: Check $\sum_{k=0}^{\infty} |a_k| \geq 0$ conv. or not
(using Step 2)

Def

A series $\sum_{k=0}^{\infty} a_k$ is absolutely convergent if $\sum_{k=0}^{\infty} |a_k|$ is convergent.

A series $\sum_{k=0}^{\infty} a_k$ is conditionally convergent if $\begin{cases} \sum_{k=0}^{\infty} a_k \text{ conv. and} \\ \sum_{k=0}^{\infty} |a_k| \text{ div.} \end{cases}$

Thm (Thm 12.5.1)

Absolutely convergent series are
Convergent.

pf

Assume $\sum_{k=0}^{\infty} |a_k|$ conv.

Consider $\sum_{k=0}^{\infty} (a_k + |a_k|)$

Note that $a_k + |a_k| \geq 0 \quad \forall k$

and $a_k + |a_k| \leq |a_k| + |a_k| = 2|a_k|$

Since $\sum_{k=0}^{\infty} |a_k|$ conv., we have

$$2 \sum_{k=0}^{\infty} |a_k| = \sum_{k=0}^{\infty} 2|a_k|$$

Conv. as well
Basic Comparison

$\Rightarrow$ Test

$$\sum_{k=0}^{\infty} (a_k + |a_k|) \text{ conv.}$$

$$\Rightarrow \sum_{k=0}^{\infty} (a_k + |a_k|) - \sum_{k=0}^{\infty} |a_k| \stackrel{\text{also}}{\text{conv.}}$$

$=$

$$\sum_{k=0}^{\infty} (a_k + \cancel{|a_k|}) - \cancel{|a_k|} = \sum_{k=0}^{\infty} a_k \text{ conv.}$$

#

Example

$$\textcircled{1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$$

is absolutely conv. because

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ conv. (p-series } p=2>1 \text{)}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \text{ conv.}$$

$$\textcircled{2} \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} + \frac{1}{2^5} - \frac{1}{2^6} + \dots$$

Is absolutely conv. because

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \text{ conv. } (\Rightarrow \text{This series conv.})$$

Step 3-2 What if $\sum |a_k|$ div. ?

e.g. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} \text{ div. (p-series } p=1 \leq 1)$$

We can have the following test:

Thm (Alternating Series Test, Thm 12.5.3)

Let $(a_k)_{k=1}^{\infty}$ be a decreasing, positive seq. Then if $\lim_{k \rightarrow \infty} a_k = 0$, then

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \dots$$

is convergent.

Example

Since $a_k = \frac{1}{k}$ is decreasing, positive

and $\lim_{k \rightarrow \infty} \frac{1}{k} = 0$, by Alternating

Series Test, we know

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

is (conditionally) convergent.

Remark

A rearrangement of $\sum_{k=1}^{\infty} a_k$ is a series that has exactly same terms but in a different order.

For example,

$$a+b = b+a \quad \begin{matrix} 1+2=2+1 \\ =3 \end{matrix}$$

$$a_1 + a_3 + a_5 + a_7 + a_2 + a_9 + a_{11} + a_{13} + a_{15} + a_4 + \dots$$

is a rearrangement of $\sum_{k=1}^{\infty} a_k$

In fact, we know:

- (i) If $\sum_{k=1}^{\infty} |a_k|$ conv., then all the arrangements of $\sum_{k=1}^{\infty} a_k$ still converge absolutely to the same limit $\sum_{k=1}^{\infty} a_k$
- (ii) If $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, a rearrangement of $\sum_{k=1}^{\infty} a_k$ can conv. to a different limit.
In fact, it can be rearranged to

Converge to ANY real number,
or diverge to $+\infty$, to $-\infty$,
or even to oscillate between

any two numbers we choose.

why? If $\sum_{k=1}^{\infty} a_k$ is conditionally conv,
then $\exists \infty$ terms < 0 and
 ∞ terms > 0

Assume $a_1^+, a_2^+, a_3^+, \dots$ are the positive terms
 $a_1^-, a_2^-, a_3^-, \dots$ are negative terms

If we want to rearrange $\sum a_k$ to
conv. to 100, we can do:

$$\underbrace{a_1^+ + a_2^+ + \dots + a_{k_1}^+}_{\geq 100} + \underbrace{a_1^- + a_2^- + \dots + a_{k_2}^-}_{\leq 100} + a_{k_1+k_2+1}^+ + \dots$$

and $a_1^+ + \dots + a_{k_1-1}^+ < 100$

Taylor expansion

We have understood the differentiation
and integration of polynomials:

- $(a_0 + a_1x + a_2x^2 + \dots + a_nx^n)'$

$$= a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}$$

$$\bullet \int_C^d (a_0 + a_1x + a_2x^2 + \dots + a_nx^n) dx$$

$$= \left(a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \dots + \frac{a_n}{n+1}x^{n+1} \right) \Big|_{x=C}^d$$

Now we try to approach more complicated functions by polynomials. For

$$f(x) \approx \boxed{a_0 + a_1x + a_2x^2 + \dots + a_nx^n} \quad \text{poly}$$

let us assume they have the same derivatives at 0 up to n-th order:

$$f^{(0)}(0) = \underline{f(0)} = p^{(0)}(0) = p(0) = \underline{a_0}$$

$$f^{(1)}(0) = \underline{f'(0)} = p'(0) = \underline{a_1}$$

$$f^{(2)}(0) = \underline{f''(0)} = p''(0) = \underline{2a_2}$$

⋮

$$f^{(n)}(0) = p^{(n)}(0) = n! \cdot a_n$$

So we consider the polynomial

$$p(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

As the "best" degree- n polynomial approximation of $f(x)$ around $x=0$.

If we choose a larger n , then we get a better approximation of $f(x)$.

Thus, to optimize approximation of this type, we consider the series approximation:

$$\begin{aligned} f(x) &\approx \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \\ &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots \end{aligned}$$

"Taylor expansion at $x=0$ "

A couple of natural questions arise:

(i) For what x , does the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ Converge?

(ii) If the series Conv. at x , does it really converge to $f(x)$?