

Calculus, week 3, Fall 2026

Recall

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx$$

Example

① $\int_1^\infty \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-2} dx$ " x^{-2} " " $(-1)x^{-1}$ "

Recall
• $F(x) = f(x) \Rightarrow \int_a^b f(x) dx = F(b) - F(a)$
• $(x^r)' = r \cdot x^{r-1}$

$$= \lim_{b \rightarrow \infty} \left(-x^{-1} \Big|_1^b \right) = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right) = 1$$

② $\int_1^\infty \frac{1}{\sqrt{x}} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-\frac{1}{2}} dx$ " $x^{-\frac{1}{2}}$ " " $(2x^{\frac{1}{2}})'$ "

$$= \lim_{b \rightarrow \infty} \left(2x^{\frac{1}{2}} \Big|_1^b \right) = \lim_{b \rightarrow \infty} (2\sqrt{b} - 2) \text{ diverges}$$

③ Last week, we computed a more general improper integral:

$$\int_1^\infty \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p < 1 \end{cases}$$

Now we want to compute for $p=1$:

$$\int_1^{\infty} \frac{1}{x} dx = ?$$

Sol

Step 1 $\int_1^b \left(\frac{1}{x} \right) dx = ?$

Notation:

$$F(b) - F(a) = F(x) \Big|_a^b$$

Recall: $(\ln x)' = \frac{1}{x}$ for $x > 0$

$$\begin{aligned} \int_1^b \frac{1}{x} dx &= \ln x \Big|_1^b \\ &= \ln b - \ln 1 = \ln b \end{aligned}$$

Step 2 Take limit

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx$$

by Step 1

$$= \lim_{b \rightarrow \infty} \ln b \quad \text{diverges}$$

So $\int_1^{\infty} \frac{1}{x} dx$ diverges. #

Conclusion

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

Example

$$\textcircled{1} \int_1^{\infty} \frac{\ln x}{x^2} dx = ?$$

Sol

Step 1

$$\int_1^b \frac{\ln x}{x^2} dx = ?$$

Let

$$u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x}$$

$$v(x) = -x^{-1}$$

$$v'(x) = x^{-2}$$

Then

$$\int_1^b \frac{\ln x}{x^2} dx = \int_1^b \underbrace{u(x)}_{\ln x} \cdot \underbrace{v'(x)}_{x^{-2}} dx$$

$$\stackrel{\textcircled{2}}{=} (\ln x) \cdot (-x^{-1}) \Big|_1^b$$

$$- \int_1^b \frac{1}{x} \cdot (-x^{-1}) dx$$

$$= -\frac{\ln b}{b} + \frac{\ln 1}{1} + \int_1^b x^{-2} dx$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1$$

Step 2 Take limit

Recall "變數變換"

① Assume $F'(x) = f(x)$

$$\int_a^b \underbrace{f(u(t))}_{f(u)} \cdot \underbrace{u'(t) dt}_{du}$$

$$= \int_{u(a)}^{u(b)} f(u) du$$

$$\left(\text{since } \frac{d}{dt} F(u(t)) = F'(u(t)) \cdot u'(t) \right. \\ \left. = f(u(t)) \cdot u'(t) \right)$$

$$\textcircled{2} \int_a^b u(t) v'(t) dt$$

$$= u(t) v(t) \Big|_a^b$$

$$- \int_a^b v(t) u'(t) dt$$

"分部積分"

$$-x^{-1} \Big|_1^b = -\frac{1}{b} + 1$$

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{b \rightarrow \infty} \left(-\frac{\ln b}{b} - \frac{1}{b} + 1 \right)$$

$\lim_{b \rightarrow \infty} \frac{\ln b}{b} \stackrel{L'H}{=} \lim_{b \rightarrow \infty} \frac{1/b}{1} = 0$

$$= 1$$

$$\textcircled{2} \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = ?$$

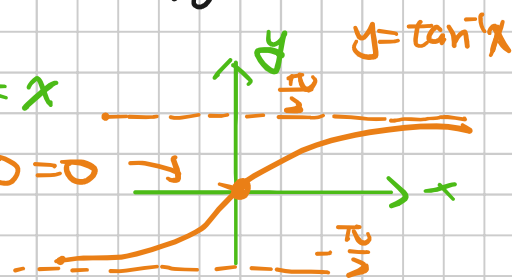
Sol

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx$$

Recall

$$\tan^{-1} x = y \Leftrightarrow \tan y = x$$

$$\tan^{-1} 0 = 0$$



$$(\tan^{-1} x)' = \frac{1}{1+x^2}$$

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx$$

$$= \lim_{b \rightarrow \infty} (\tan^{-1} x) \Big|_0^b = \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0)$$

$$= \lim_{b \rightarrow \infty} \tan^{-1} b = \frac{\pi}{2}$$

$$\int_{-\infty}^0 \frac{1}{1+x^2} dx = \lim_{a \rightarrow -\infty} \int_a^0 \frac{1}{1+x^2} dx$$

$= (\tan^{-1} x)$

$$= \lim_{a \rightarrow -\infty} (\tan^{-1} 0 - \tan^{-1} a)$$

$$= - \lim_{a \rightarrow -\infty} \tan^{-1} a = - \left(-\frac{\pi}{2} \right) = \frac{\pi}{2}$$

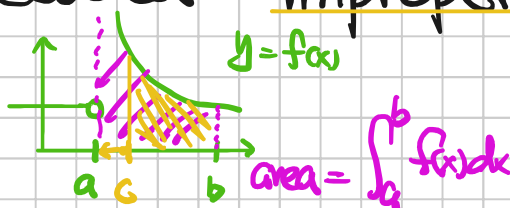
$$\text{So } \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

~~≠~~

Def

Integrals of functions that become $\pm\infty$ at a point in the interval of integration are called improper integrals of type II



ii) If $f(x)$ is continuous on $[a, b]$ and discontinuous at a , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

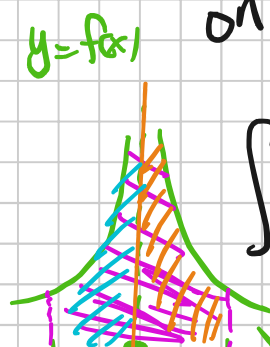
($a < c$)

ii) If $f(x)$ is continuous on $[a, b)$ and discontinuous at b , then



$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

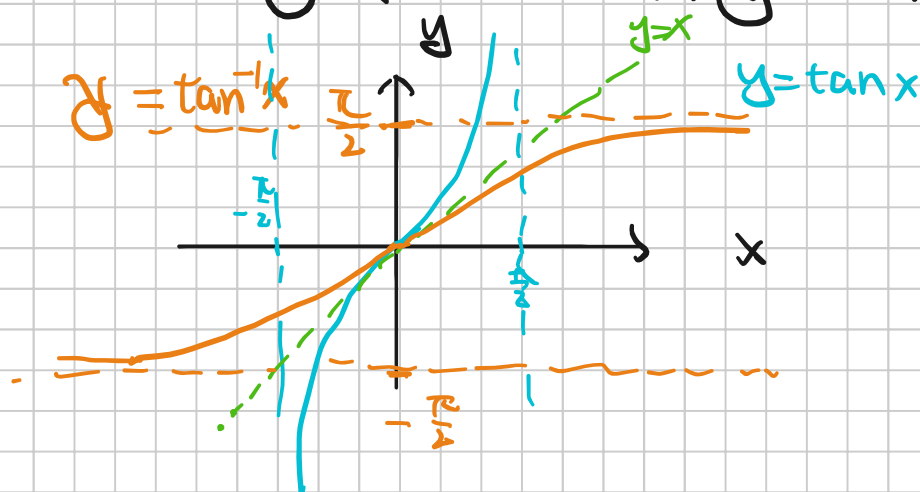
iii) If $f(x)$ is discontinuous at c , where $a < c < b$, and continuous on $[a, c)$ and on $(c, b]$, then



$$\int_a^b f(x) dx = \underbrace{\int_a^c f(x) dx}_{(ii)} + \underbrace{\int_c^b f(x) dx}_{(i)}$$

Review

① $\tan^{-1} x = y \Leftrightarrow \tan y = x$



② $(\tan^{-1} x)' = ?$

Recall the chain rule:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Using the chain rule, we compute:

- $(\tan(\tan^{-1}x))' = (x)' = 1$

$f(y) = \tan y$
 $g(x) = \tan^{-1}x$

$\rightarrow //$

$$\tan'(\tan^{-1}x) \cdot (\tan^{-1}x)'$$

- $\tan x = \frac{\sin x}{\cos x}$, $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

$$\left(\frac{b(x)}{a(x)}\right)' = \frac{b'(x)a(x) - b(x)a'(x)}{(a(x))^2}$$

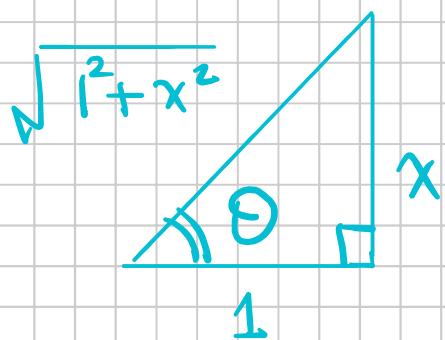
$$\Rightarrow (\tan x)' = \frac{\overset{\cos x}{(\sin x)'} \cos x - \sin x \cdot \overset{-\sin x}{(\cos x)'}}{(\cos x)^2}$$

$$= \frac{1}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$(\tan x)' = \sec^2 x$$

- $(\tan y)' \big|_{y=\tan^{-1}x} = \sec^2 y \big|_{y=\tan^{-1}x}$

$$= \sec^2(\tan^{-1} x) = 1 + x^2$$



$$\tan \theta = x$$

$$\theta = \tan^{-1} x$$

$$\Rightarrow \sec \theta = \frac{\sqrt{1+x^2}}{1}$$

• Since

$$\left((\tan y)' \right)_{y=\tan^{-1} x} \cdot (\tan^{-1} x)' = 1$$

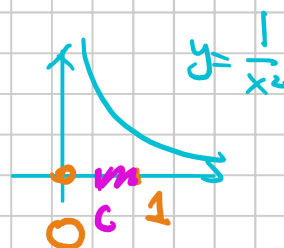
we conclude that

$$(\tan^{-1} x)' = \frac{1}{1+x^2} \quad \#$$

$$\Leftrightarrow \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

Example

① $\int_0^1 \frac{1}{x^2} dx = ?$



Sol

$\frac{1}{x^2}$ is discontinuous at 0

$$\begin{aligned}
 \int_0^1 \frac{1}{x^2} dx &\stackrel{w}{=} \lim_{c \rightarrow 0^+} \int_c^1 \left(\frac{1}{x^2} \right) dx \\
 &\stackrel{w}{=} \lim_{c \rightarrow 0^+} \left(-x^{-1} \Big|_c^1 \right) \\
 &= \lim_{c \rightarrow 0^+} \left(-\frac{1}{1} - \left(-\frac{1}{c} \right) \right) = \lim_{c \rightarrow 0^+} \left(\frac{1}{c} - 1 \right)
 \end{aligned}$$

∞
 $\uparrow$

diverges. #

② $\int_0^1 \frac{1}{\sqrt{x}} dx = ?$

Sol

NOT continuous at $x=0$

$$\begin{aligned}
 \int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{c \rightarrow 0^+} \int_c^1 \left(\frac{1}{\sqrt{x}} \right) dx \\
 &= \lim_{c \rightarrow 0^+} \left(2\sqrt{x} \Big|_c^1 \right) = \lim_{c \rightarrow 0^+} \left(2 - 2\sqrt{c} \right)
 \end{aligned}$$

$= 2$ #

③ $\int_0^1 \frac{1}{x^p} dx = ?$

$\left(\begin{array}{l} \textcircled{1}: p=2 \\ \textcircled{2}: p=\frac{1}{2} \end{array} \right)$

sol

case 1: $P > 1$

$$\int_0^1 \frac{1}{x^p} dx = \lim_{C \rightarrow 0^+} \int_C^1 \frac{1}{x^p} dx$$

diverges if $P > 1$

$$= \lim_{C \rightarrow 0^+} \left(\frac{x^{-p+1}}{-p+1} \Big|_{x=C}^1 \right)$$

$$x^{-p} = \left(\frac{1}{-p+1} x^{-p+1} \right)'$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C \quad (n \neq -1)$$

$$= \lim_{C \rightarrow 0^+} \left(\frac{1}{-p+1} - \frac{C^{-p+1}}{-p+1} \right)$$

$$= \lim_{C \rightarrow 0^+} \left(\frac{1}{-p+1} - \frac{1}{-p+1} \cdot \frac{1}{C^{p-1}} \right)$$

"1/0"

which diverges.

case 2: $P < 1$

$$\int_0^1 \frac{1}{x^p} dx = \lim_{C \rightarrow 0^+} \int_C^1 x^{-p} dx$$
$$= \lim_{C \rightarrow 0^+} \left(\frac{x^{-p+1}}{-p+1} \Big|_C^1 \right)$$
$$= \frac{1}{-p+1}$$

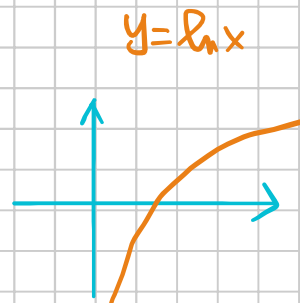
Case 3 : $p = 1$

$$\int_0^1 \frac{1}{x} dx = \lim_{c \rightarrow 0^+} \int_c^1 \frac{1}{x} dx = \lim_{c \rightarrow 0^+} (\ln x)' \Big|_c^1$$

$$= \lim_{c \rightarrow 0^+} (\ln 1 - \ln c)$$

$-\infty$

diverges



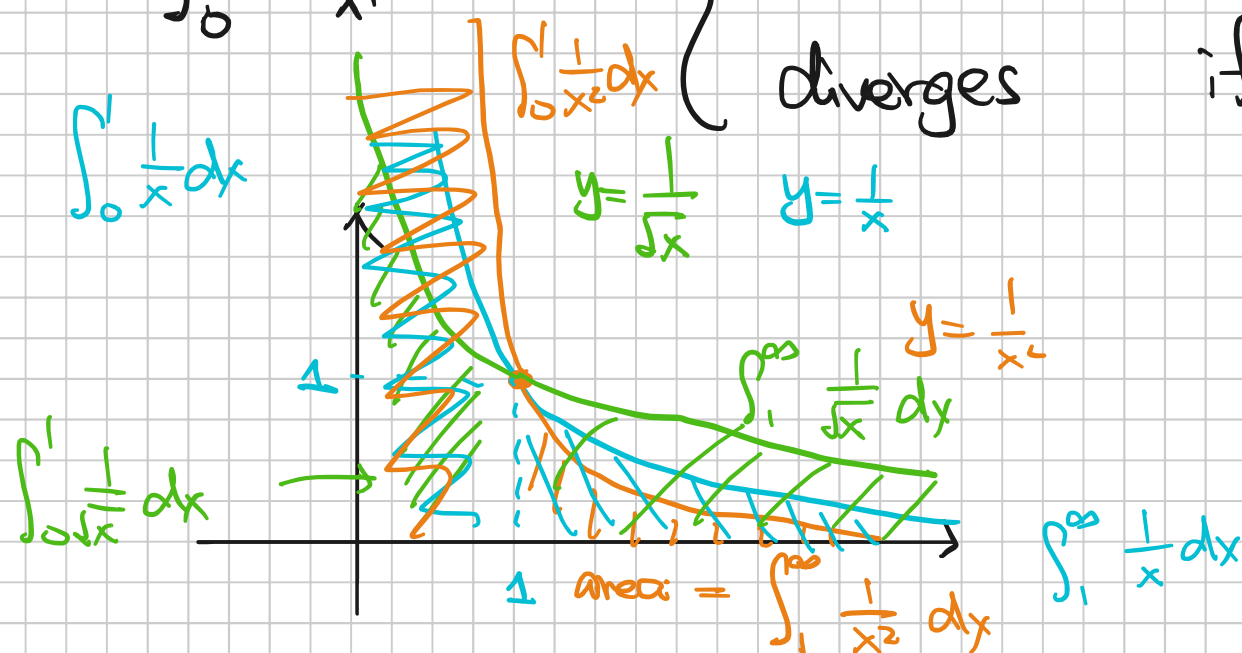
Conclusion

$$\int_0^1 \frac{1}{x^p} dx = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}$$

if $p < 1$

if $p \geq 1$

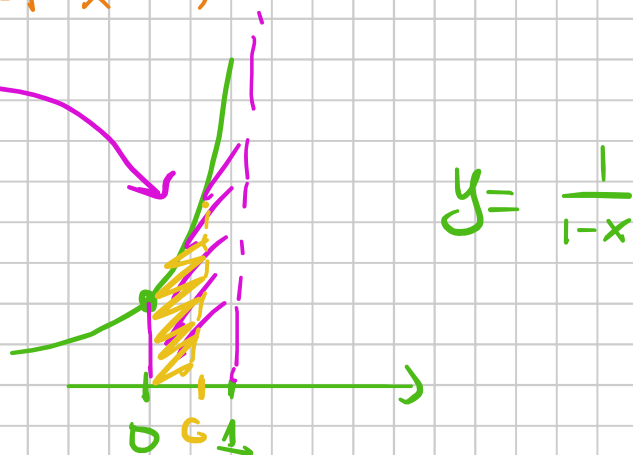
#



Example

① $\int_0^1 \frac{1}{1-x} dx = ?$

Sol



$$\int_0^1 \frac{1}{1-x} dx = \lim_{c \rightarrow 1^-} \int_0^c \frac{1}{1-x} dx$$

Step 1: Compute $\int_0^c \frac{1}{1-x} dx$ Find $F(x)$ s.t.
 $= F'(x)$ ←

Recall that

- $(\ln x)' = \frac{1}{x}$ (for $x > 0$)

- $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

- Take $f(y) = \ln y$, $g(x) = 1-x$

$$\Rightarrow (f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{g(x)} \cdot g'(x) = \frac{1}{1-x} \cdot (1-x)'$$

$$= \frac{-1}{1-x}$$

So if we set

$$F(x) = -f(g(x)) = -\ln(1-x)$$

then

$$F'(x) = - \frac{(1-x)'}{1-x} = \frac{1}{1-x}$$

$$\Rightarrow \int_0^c \frac{1}{1-x} dx = F(c) - F(0) = -\ln(1-c) - (-\ln(1-0)) = 0$$

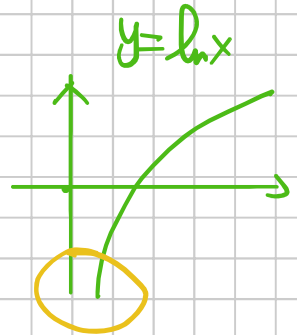
$$= -\ln(1-c)$$

Step 2 Take limit:

$$\int_0^1 \frac{1}{1-x} dx = \lim_{c \rightarrow 1^-} \int_0^c \frac{1}{1-x} dx$$

$$= \lim_{c \rightarrow 1^-} (-\ln(1-c))$$

$\downarrow$ $+\infty$ $\downarrow$ 0

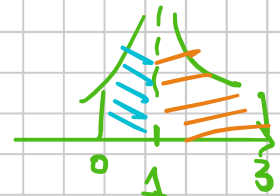


diverges

#

② $\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx = ?$

NOTE:
 $(x-1)^{\frac{2}{3}} = 0$ when $x=1$



sol

$$\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx = \int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}} dx + \int_1^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx$$

Term I Term II

Term I:

$$\int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}} dx = \lim_{c \rightarrow 1^-} \int_0^c \frac{1}{(x-1)^{\frac{2}{3}}} dx$$

$= (x-1)^{-\frac{2}{3}}$

NOTE:

• $(x^r)' = r x^{r-1}$

$f'(y) = \frac{1}{3} \cdot y^{-\frac{2}{3}}$

• $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$g'(x) = 1$

• Take $f(y) = y^{-\frac{2}{3}+1} = \frac{1}{3}$ $g(x) = x-1$

$$\Rightarrow (f(g(x)))' = f'(x-1) \cdot (x-1)'$$

$$= \frac{1}{3} (x-1)^{-\frac{2}{3}} \cdot 1$$

So if we set

$$F(x) = 3 \cdot (x-1)^{\frac{1}{3}}$$

then

$$F'(x) = (x-1)^{-\frac{2}{3}}$$

$$\Rightarrow \int_0^c \frac{1}{(x-1)^{\frac{2}{3}}} dx = F(c) - F(0)$$

$$= 3(c-1)^{\frac{1}{3}} - 3(0-1)^{\frac{1}{3}} = 3 \cdot \sqrt[3]{-1} = -3$$

$$= 3(c-1)^{\frac{1}{3}} + 3$$

$$\Rightarrow \int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}} dx = \lim_{c \rightarrow 1^-} 3\sqrt[3]{c-1} + 3 = 3$$

Term II:

$$\int_1^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx = \lim_{c \rightarrow 1^+} \int_c^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx$$

$$= \lim_{c \rightarrow 1^+} (F(3) - F(c))$$

$$F(x) = 3\sqrt[3]{x-1}$$

$$3\sqrt[3]{1-1} = 0$$

$$= \lim_{c \rightarrow 1^+} \left(3 \sqrt[3]{3-c} - 3 \sqrt[3]{c-1} \right)$$

$$= 3 \sqrt[3]{2}$$

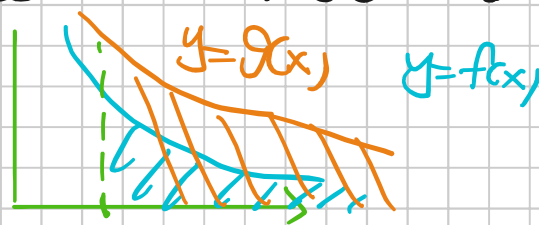
Conclusion

$$\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}} dx = \text{Term I} + \text{Term II}$$

$$= 3 + 3 \sqrt[3]{2} \quad \#$$

Q: How do we know an improper integral converges or not?

Thm (§ 11.7)



Let f and g be continuous on $[a, \infty)$ with

$$0 \leq f(x) \leq g(x) \quad \forall x \in [a, \infty)$$

Then

$$(i) \int_a^{\infty} g(x) dx \text{ Converges}$$

$$\Rightarrow \int_a^{\infty} f(x) dx \text{ Converges}$$

(ii) $\int_a^\infty f(x) dx$ diverges

$\Rightarrow \int_a^\infty g(x) dx$ diverges

Thm

Assume f and g are positive continuous functions on $[a, \infty)$, and

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty.$$

(f and g grow at the same rate)

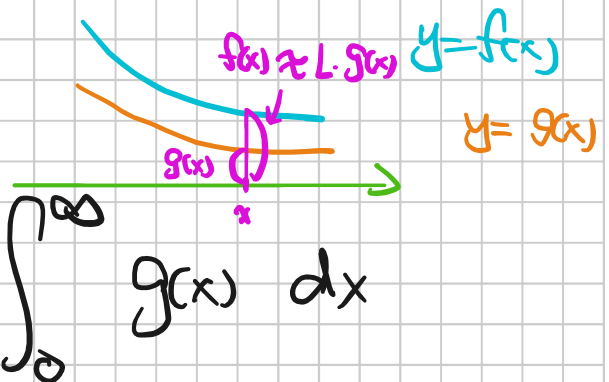
Then

$$\int_a^\infty f(x) dx \quad \text{and} \quad \int_a^\infty g(x) dx$$

either both converges or
both diverges.

Example

Determine the following integrals converge or diverge



$$\textcircled{1} \int_1^{\infty} \frac{\sin^2 x}{x^2} dx$$

Compare it with $\frac{1}{x^2}$

Recall

$$\int_1^{\infty} \frac{1}{x^p} dx \begin{cases} \text{Converges} & \text{for } p > 1 \\ \text{diverges} & \text{for } p \leq 1 \end{cases}$$

Since

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \quad \forall x \in [1, \infty)$$

and

$$\int_1^{\infty} \frac{1}{x^2} dx \text{ Converges}$$

We conclude that

$$\int_1^{\infty} \frac{\sin^2 x}{x^2} dx \text{ Converges.}$$

#

$$\textcircled{2} \int_1^{\infty} \frac{1}{\sqrt{x^2 - \frac{1}{2}}} dx \sim \frac{1}{x}$$

method I

$$\frac{1}{\sqrt{x^2 - \frac{1}{2}}} \geq \frac{1}{\sqrt{x^2}} = \frac{1}{x} \geq 0 \quad \forall x \geq 1$$

and

$$\int_1^{\infty} \frac{1}{x} dx \text{ diverges}$$

So

$$\int_1^{\infty} \frac{1}{\sqrt{x^2 - \frac{1}{2}}} dx \text{ diverges}$$

#

method II

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{\frac{1}{\sqrt{x^2 - \frac{1}{2}}}}{\frac{1}{x}} &= \lim_{x \rightarrow \infty} \frac{x \cancel{/x}}{\sqrt{x^2 - \frac{1}{2}} \cancel{/x}} \\&= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{\frac{x^2}{x^2} - \frac{1}{2} \cdot \frac{1}{x^2}}} = \frac{1}{\sqrt{1}} = 1 \neq 0 \neq \infty\end{aligned}$$

And $\int_1^{\infty} \frac{1}{x} dx$ diverges

So $\int_1^{\infty} \frac{1}{\sqrt{x^2 - \frac{1}{2}}} dx$ diverges $\neq$

③ $\int_1^{\infty} \frac{1}{1+x^2} dx \sim \frac{1}{x^2}$

Since

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{x^2 \cancel{/x^2}}{1+x^2 \cancel{/x^2}} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x^2} + 1}$$

and $\int_1^{\infty} \frac{1}{x^2} dx = 1 \neq 0 \neq \infty$ $(\frac{1}{x^2} < \frac{1}{x})$ Converges

we have

$$\int_1^{\infty} \frac{1}{1+x^2} dx \quad \text{Converges} \quad \#$$

$$\textcircled{1} \int_1^{\infty} \frac{1-e^{-x}}{x} dx \quad \sim \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} \frac{\frac{1-e^{-x}}{x}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} 1 - e^{-x} = 1 \quad \begin{matrix} \nearrow 0 \\ \nwarrow \infty \end{matrix}$$

and $\int_1^{\infty} \frac{1}{x} dx$ diverges

So $\int_1^{\infty} \frac{1-e^{-x}}{x} dx$ diverges. $\#$

§ Series 級數

In this section, we consider "infinite sum"

$$a_0 + a_1 + a_2 + a_3 + \dots$$

Basic question:

Do we really get a number by an infinite sum?

Def

Let $(a_n)_{n=0}^{\infty}$ be a sequence. Consider

$$\left(\sum_{k=0}^n a_k \right)_{n=0}^{\infty}$$

$$= a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots$$

(the seq of partial sums)

We say the series $\sum_{k=0}^{\infty} a_k$ Converges to L if the seq.

$$\left(\sum_{k=0}^n a_k \right)_{n=0}^{\infty}$$

Converges to L:

$$\sum_{k=0}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k = L$$

The number L is called the sum of the series $\sum_{k=0}^{\infty} a_k$.

If $\left(\sum_{k=0}^n a_k \right)_{n=0}^{\infty}$ diverges, then we say the $\sum_{k=0}^{\infty} a_k$ diverges.

Example

$$\textcircled{1} \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = ?$$

Sol

Since

NOTE:

$$\frac{(k+2) - (k+1)}{(k+1)(k+2)} = \frac{\cancel{(k+2)}}{(k+1)\cancel{(k+2)}} - \frac{\cancel{(k+1)}}{\cancel{(k+1)}(k+2)}$$

$$= \frac{1}{k+1} - \frac{1}{k+2}$$

$$\sum_{k=0}^n \frac{1}{(k+1)(k+2)} = \sum_{k=0}^n \left(\frac{1}{k+1} - \frac{1}{k+2} \right)$$

$$= \left(\frac{1}{0+1} - \frac{1}{0+2} \right) + \left(\frac{1}{1+1} - \frac{1}{1+2} \right) + \left(\frac{1}{2+1} - \frac{1}{2+2} \right) + \dots + \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

$$= 1 - \frac{1}{n+2}$$

we have

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{(k+1)(k+2)}$$

$$= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+2} \right) = 1 \quad \#$$

$$\textcircled{2} \sum_{k=0}^{\infty} (-1)^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n (-1)^k$$

Since

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 \dots$$

$$= \begin{cases} 0 \\ 1 \end{cases}$$

n is even

n is odd

diverges, the series

$$\sum_{k=0}^{\infty} (-1)^k$$

diverges

#

(3) (Geometric series)

$$\sum_{k=0}^{\infty} x^k = ?$$

$$a + ar + ar^2 + \dots + ar^n$$

$$= a \frac{1 - r^{n+1}}{1 - r}$$

Since

$$(1-x)(1+x+x^2+\dots+x^n) = 1 - x^{n+1}$$

$$\sum_{k=0}^n x^k = 1 + x + x^2 + \dots + x^n$$

$$\frac{1 - x^{n+1}}{1 - x}$$

if $|x| < 1$
if $|x| > 1$ or $x = -1$
if $x \neq 1$

if $x = 1$

$(n+1)$ diverges

we have

$$\sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x} \\ \text{diverges} \end{cases}$$

if $|x| < 1$

if $|x| \geq 1$ #