Calculus, week 2, Fall 2026

Thm (Pinching Thm, Thm 11.3.9)

If $a_n \le b_n \le c_n$ for n sufficiently large, and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$$

then $$\lim_{n\to\infty} b_n = L.$$

Example

① $$\lim_{n\to\infty} \frac{\cos n}{n} = 0,$$ because

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

and $$\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$$

② $$\lim_{n\to\infty} \sqrt{4+\left(\frac{1}{n}\right)^2} = 2$$

$(a+b)^2 = a^2+b^2+2ab$

$a=2$, $b=\frac{1}{n}$

because

$$4 \le 4+\left(\frac{1}{n}\right)^2 \le \underbrace{\underset{\underset{2^2}{\shortparallel}}{4}+\left(\frac{1}{n}\right)^2+2\cdot 2\cdot\frac{1}{n}}_{}$$

$= \left(2+\frac{1}{n}\right)^2$

$$\Rightarrow \quad \underset{\substack{\shortparallel \\ 2}}{\sqrt{4}} \leq \sqrt{4+\left(\frac{1}{n}\right)^2} \leq \underset{\substack{\shortparallel \\ 2+\frac{1}{n}}}{\sqrt{\left(2+\frac{1}{n}\right)^2}}$$

and $\lim\limits_{n\to\infty} 2 = 2 = \lim\limits_{n\to\infty}\left(2+\frac{1}{n}\right)$

$$\Rightarrow \quad \lim_{n\to\infty} \sqrt{4+\left(\frac{1}{n}\right)^2} = 2 \qquad \#$$

<u>Thm</u> (Thm 11.3.12)

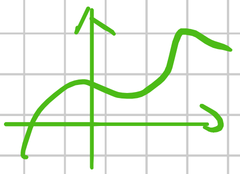
If $\lim\limits_{n\to\infty} c_n = c$ and $f = f(x)$ is <u>continuous</u> at c, then

連續

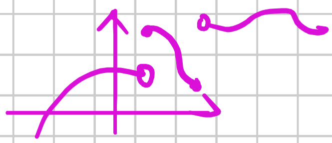
$$\lim_{n\to\infty} f(c_n) = f(c) = f\left(\lim_{n\to\infty} c_n\right)$$

<u>Recall</u>

The graph of a continuous function looks like

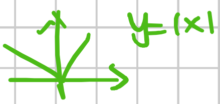


$\leftrightarrow$

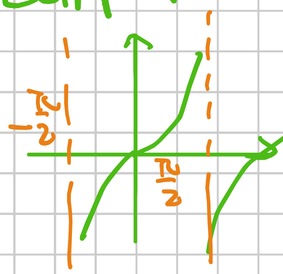


NOT continuous

The following functions are continuous:

- polynomials: $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
- $\sin x$, $\cos x$, $|x|$ 

- e^x , a^x , $xe^x + \cos x \cdot \sin x + 2^x \cdot (x^2+1) + e^{\sin x}$
- $\log x$, $\ln x$ $\qquad \underline{\underline{x > 0}}$
- $\sqrt{x}$, $\sqrt[n]{x}$ $\qquad x > 0$
- $\tan x = \dfrac{\sin x}{\cos x}$ $\qquad \underline{\underline{-\frac{\pi}{2} < x < \frac{\pi}{2}}}$ $\Rightarrow \cos x \neq 0$



Example

(2) $\displaystyle\lim_{n\to\infty} \sqrt{4 + \left(\tfrac{1}{n}\right)^2} = ?$

method II :

Consider $\quad f(x) = \sqrt{4 + x^2}$ $\Leftarrow$ polynomial ≥ 0

$\quad f(x)$ is continuous

Let $c_n = \frac{1}{n} \Rightarrow \displaystyle\lim_{n\to\infty} c_n = 0$

$\Rightarrow \quad f(c_n) = \sqrt{4 + \left(\tfrac{1}{n}\right)^2}$

$\Rightarrow$ ans $= \displaystyle\lim_{n\to\infty} f(c_n) = f\Big(\underbrace{\lim_{n\to\infty} c_n}_{=0}\Big)$

$= f(0) = \sqrt{4 + 0^2} = \sqrt{4} = 2$ #

(3) $\displaystyle\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) \overset{\sin x \text{ is continuous}}{=} \sin\Big(\underbrace{\lim_{n\to\infty} \frac{\pi}{n}}_{=0}\Big) = \sin 0 = 0$ #

$e = 2, \cdots$

$\ln = \log_e$

$(e^x)' = e^x$

$(\ln x)' = \frac{1}{x}$ $(x > 0)$

$$\lim_{n\to\infty} e^{\frac{\pi}{n}} = e^{\lim_{n\to\infty}\frac{\pi}{n}} = e^0 = 1 \quad \#$$

e^x is continuous

tan x is continuous for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\lim_{n\to\infty} \tan\left(\frac{\pi}{n}\right) = \tan\left(\lim_{n\to\infty}\frac{\pi}{n}\right) = \tan 0 = 0 \quad \#$$

④ |x| is continuous

$$\lim_{n\to\infty}\left|\frac{2n-1}{n}\right| = \left|\lim_{n\to\infty}\frac{(2n-1)\cdot\frac{1}{n}}{(n)\cdot\frac{1}{n}}\right|$$

$$= |2| = 2$$

$$\lim_{n\to\infty}\frac{2\left(1-\frac{1}{n}\right)}{1} = 2 \quad \#$$

$$\lim_{n\to\infty}\sqrt{\frac{2n-1}{n}} = \sqrt{\lim_{n\to\infty}\frac{2n-1}{n}} = \sqrt{2} \quad \#$$

$$\lim_{n\to\infty}\ln\left(\frac{2n-1}{n}\right) = \ln\left(\lim_{n\to\infty}\frac{2n-1}{n}\right) = \ln 2 \quad \#$$

Thm (L'Hôpital's Rule)

Let f, g be differentiable functions.

Suppose both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are 0 or $\pm\infty$ ("$\frac{0}{0}$" "$\frac{\infty}{\infty}$")

Then $$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)}$$

Example

① $$\lim_{n\to\infty} \frac{\ln n}{n} = ?$$

sol

Consider the limit of function:

" $\frac{\infty}{\infty}$ " $$\lim_{x\to\infty} \frac{\ln x \to \infty}{x \to \infty}$$

By L'Hôpital,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{(\ln x)'}{(x)'}$$

Recall:

$(\ln x)' = \frac{1}{x}$

$(x)' = 1$

$(x^n)' = n \cdot x^{n-1}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\ln n}{n} = 0 \quad \#$$

② $$\lim_{n\to\infty} n(e^{\frac{1}{n}} - 1) = ?$$

sol

Consider the limit of function:

$$\lim_{x\to\infty} \overset{\infty}{x} \cdot \left(e^{\frac{1}{x}}-1\right) \to e^{0}-1=0$$

" $\infty \cdot 0$ " $\overset{"}{=}$ "

$\frac{1}{0}\cdot 0 = \frac{0}{0}$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}}-1 \to 0}{\frac{1}{x} \to 0} \quad \text{" } \frac{0}{0} \text{ "}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\boxed{\left(e^{\frac{1}{x}}-1\right)'}}{\left(\frac{1}{x}\right)'}$$

$\left(\frac{1}{x}\right)' = \frac{-1}{x^2}$

$\left(f(g(x))\right)'$

$= f'(g(x)) \cdot g'(x)$

$= e^{g(x)} \cdot \left(\frac{-1}{x^2}\right)$

$= e^{\frac{1}{x}} \cdot \left(\frac{-1}{x^2}\right)$

Recall:

$\left(\frac{1}{g(x)}\right)' = \frac{-g'(x)}{(g(x))^2}$

$\left(\frac{1}{x}\right)' = \frac{-(x)'}{x^2} = \frac{-1}{x^2}$

$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$

$\boxed{f(y) = e^{y}-1}$, $g(x) = \frac{1}{x}$

$\Rightarrow f'(y) = \underbrace{(e^{y})'}_{= e^{y}} - \underbrace{(1)'}_{=0}$

$= e^{y}$

$$= \lim_{x\to\infty} \frac{e^{\frac{1}{x}} \cdot \cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}} = \lim_{x\to\infty} e^{\frac{1}{x}}$$

$$= e^{\lim_{x\to\infty} \frac{1}{x}} = e^{0} = 1.$$

$$\Rightarrow \lim_{n\to\infty} n\left(e^{\frac{1}{n}}-1\right) = 1$$

#

Thm (§11.4)

(i) $\lim_{n\to\infty} a^{\frac{1}{n}} \left(= a^{\lim_{n\to\infty}\frac{1}{n}} = a^0 = 1\right) = 1$ for $a>0$

(ii) $\lim_{n\to\infty} a^n = 0$ for $|a|<1$

e.g. $a=\frac{1}{2}$ $\quad (\frac{1}{2})^n: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$

x^a is continuous $a>0$

(iii) $\lim_{n\to\infty} \frac{1}{n^a} \left(= \left(\lim_{n\to\infty}\frac{1}{n}\right)^a = 0^a\right) = 0$ for $a>0$

(iv) $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ $\quad n! = 1\times2\times3\times\cdots\times n$

$a = [a].a_1a_2a_3\cdots$

$a = 2.34 \Rightarrow [a]=2$

$0 \leftarrow \left|\frac{a^n}{n!}\right| = \left|\frac{a\times a\cdots a}{1\times2\times\cdots n}\right| = \left|\underbrace{\frac{a\cdots a}{1\cdots[a]}}_{M}\cdot\frac{a\cdots a}{([a]+1)\cdots n}\right| < 1$

$\leq |M|\cdot\frac{|a|}{n} \to 0$

(v) $\lim_{n\to\infty}\frac{\ln n}{n} = 0$

(vi) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Consider $\lim_{x\to\infty} x^{\frac{1}{x}}$

Recall: $\begin{cases} e^{\ln y} = y & y>0 \\ \ln e^z = z & z: \text{real any number} \end{cases}$

Special method: $x^{\frac{1}{x}} = e^{\ln(x^{\frac{1}{x}})}$

對數率 $= e^{\frac{1}{x}\cdot \ln x}$

e^{y} is continuous

$$\Rightarrow \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^{\boxed{\lim_{x\to\infty}\frac{\ln x}{x}}}$$

$\lim_{x\to\infty}\frac{\ln x}{x} = 0$

$$= e^{0} = 1$$

$$\Rightarrow \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{n} = 1 \quad \#$$

<u>Thm</u> (§11.4)

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$$

<u>Recall</u>: The key special property of e:

$$(e^{x})' = e^{x}$$

In particular,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \overset{x=1}{=} e = 2,\cdots$$

Pf

<u>case 1</u>: $x=0$

$$\text{LHS} = \lim_{n\to\infty}\left(1+\frac{0}{n}\right)^{n} = \lim_{n\to\infty} 1^{n} = 1$$

$$\text{RHS} = e^{0} = 1$$

//

<u>case 2</u>: $x\neq 0$.

NOTE:

$$\left(1+\frac{x}{n}\right)^{n} = e^{\ln\left(1+\frac{x}{n}\right)^{n}}$$

$$= e^{n \cdot \ln(1+\frac{x}{n})}$$

$$n \cdot \ln(1+\frac{x}{n}) = \frac{\ln(1+\frac{x}{n})}{\frac{1}{n}}$$

$$= x \cdot \frac{\ln(1+\overbrace{\frac{x}{n}}^{=h}) - \ln 1}{\underbrace{\frac{x}{n}}_{=h}}$$

Recall:

- $\ln 1 = 0$
- $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

When $f(y) = \ln y$, $y = 1$

$$(\ln y)'|_{y=1} = \lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h}$$

$$(\ln y)'|_{y=1} = \frac{1}{y}\Big|_{y=1} = 1$$

$$\Rightarrow \lim_{n\to\infty} n \cdot \ln(1+\frac{x}{n})$$

$$= \lim_{n\to\infty} x \cdot \boxed{\frac{\ln(1+\frac{x}{n}) - \ln 1}{\frac{x}{n}}}$$

$\Downarrow$

$\frac{x}{n} \to 0$

$$= x \cdot (\ln y)'|_{y=1} = x$$

$$\Rightarrow \lim_{n\to\infty} (1+\frac{x}{n})^n = \lim_{n\to\infty} e^{n \ln(1+\frac{x}{n})}$$

$$= e^{\boxed{\lim_{n\to\infty} n \ln(1+\frac{x}{n})} = x} = e^x \quad \#$$

§ Relative growth rate

Def (§7.4 page 470)

Let a_n and b_n be positive for n sufficiently large.

(i) We say a_n grows faster than b_n (as $n \to \infty$) if

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0$$

(Or say b_n grows slower than a_n)

(ii) We say a_n and b_n grow at the same rate if

$$\lim_{n\to\infty} \frac{b_n}{a_n} = L^{\neq \infty} > 0$$

Example

① e^n grows faster than n^2 (as $n \to \infty$)

because

$$\lim_{n\to\infty} \frac{n^2}{e^n} = \cdots$$

Recall
$(x^n)' = n \cdot x^{n-1}$
$(e^x)' = e^x$
$2 = 2 \cdot (1 \cdot x^{1-1}) = 2 \cdot (x^1)'$

To compute it, consider

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{“}\frac{\infty}{\infty}\text{''}}{=} \lim_{x\to\infty} \frac{(x^2)'}{(e^x)'}$$

$$= \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{“}\frac{\infty}{\infty}\text{''}}{=} \lim_{x\to\infty} \frac{(2x)'}{(e^x)'}$$

$$= \lim_{x\to\infty} \frac{2}{e^x} = 0$$

$e^x \to \infty$

$$\Rightarrow \lim_{n\to\infty} \frac{n^2}{e^n} = 0 \qquad \#$$

Remark

In general, a^n (or a^x) grows faster

a polynomial $b_k n^k + b_{k-1} n^{k-1} + \cdots + b_1 n + b_0$

(or $b_k x^k + b_{k-1} x^{k-1} + \cdots + b_1 x + b_0$)

as $n\to\infty$ (or $x\to\infty$)

$2^n = o(3^n)$ $\qquad$ $2^n = O(3^n)$

② 3^n grows faster than 2^n as $n\to\infty$

because

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0 \qquad \#$$

$\frac{2}{3} < 1$

③ n^2 grows faster than $\ln n$ as $n\to\infty$

because $\qquad \ln n = o(n^2)$

$$\lim_{n\to\infty} \frac{\ln n}{n^2} = \cdots$$

Recall

$(\ln x)' = \frac{1}{x}$

$(x>0)$

To compute it, consider

$$\lim_{x\to\infty} \frac{\ln x}{x^2} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to\infty} \frac{(\ln x)'}{(x^2)'}$$

$\ln x \to \infty$, $x^2 \to \infty$

$$= \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\ln n}{n^2} = 0 \qquad \#$$

$\ln n = o(\sqrt[k]{n})$

④ $\sqrt[k]{n}$ grows faster than $\ln n$ as $n\to\infty$

because

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt[k]{n}} = \ldots$$

Recall

- $(x^{\frac{1}{k}})' = \frac{1}{k} x^{(\frac{1}{k}-1)}$
- $(x^r)' = r \cdot x^{r-1}$
  r is a real number

To compute it, consider

$$\lim_{x\to\infty} \frac{\ln x \to \infty}{x^{1/k} \to \infty} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{=} \lim_{x\to\infty} \frac{(\ln x)'}{(x^{1/k})'}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\cancel{x}}\cdot x}{\frac{1}{k}\cdot x^{(\frac{1}{k}-1)}\cdot x} = \lim_{x\to\infty} \frac{1}{\frac{1}{k}\cdot x^{\frac{1}{k}}} \to \infty$$

$$= 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\ln n}{\sqrt[k]{n}} = 0 \qquad \#$$

Remark

fast ↑

exponential $\quad a^n \quad a$

$\qquad\qquad\quad b^n \quad \vee$
$\qquad\qquad\qquad\quad\ b$

polynomial $\quad c_k n^k + c_{k-1} n^{k-1} + \cdots c_0 \qquad c_k \neq 0$

n^{r_1} $\quad r_1$

$\quad\quad\quad\ \vee$

n^{r_2} $\quad r_2$

$0 \neq d_\ell n^\ell + d_{\ell-1} n^{\ell-1} + \cdots + d_0$ $\quad k > \ell \geq 1$

k-th root

$\sqrt[k]{n} = n^{\frac{1}{k}}$ $\quad\quad \frac{1}{k} < \frac{1}{\ell}$

$\sqrt[\ell]{n} = n^{\frac{1}{\ell}}$

slow $\downarrow$

log $\quad\quad \ln n$

Def

Assume $a_n, b_n > 0$.

We say a_n is of smaller order than b_n (as $n \to \infty$) if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, denoted by

$a_n = o(b_n)$ "a_n is little-oh of b_n"

We say a_n is at most the order of b_n (as $n \to \infty$) if $\exists$ exist M s.t.

$$\frac{a_n}{b_n} \leq M \quad \text{for } n = 1, 2, 3, \cdots$$

denoted by $a_n = O(b_n)$

"a_n is big-oh of b_n"

Remark

① $a_n = o(b_n) \iff a_n$ grows slower than b_n

② $a_n = o(b_n) \underset{\not\Leftarrow}{\Rightarrow} a_n = O(b_n)$

③ If a_n and b_n grow at the same rate, then $a_n = O(b_n)$ and $b_n = O(a_n)$

Example

① $\ln n = o(n^2)$ because $\lim_{n\to\infty} \frac{\ln n}{n^2} = 0$ #

② $n^2 = o(n^3+1)$ because

$$\lim_{n\to\infty} \frac{n^2 \times \frac{1}{n^3}}{(n^3+1) \times \frac{1}{n^3}} = \lim_{n\to\infty} \frac{\frac{1}{n} \to 0}{1+\frac{1}{n^3} \to 0} = 0$$

$\swarrow 1$

#

③ $n + \sin n = O(n)$ because

$$\frac{n+\sin n}{n} \le \frac{n+1}{n} \le \frac{n+n}{n} = 2 (=M)$$

($\sin n \le 1$, $1 \le n = 1, 2, 3 \ldots$)

#

④ $e^n + n^2 = O(e^n)$ because

$$\lim_{n\to\infty} \frac{(e^n+n^2)\cdot\frac{1}{e^n}}{(e^n)\cdot\frac{1}{e^n}} = \lim_{n\to\infty} \frac{1+\frac{n^2}{e^n}}{1}$$

$\frac{n^2}{e^n} \to 0$

$= 1 \neq 0$

$\Rightarrow$ they grow at the same rate $\Rightarrow e^n + n^2 = O(e^n)$ #

Example (Algorithm of searching)

Q: Find a word in a dictionary.
(Design algorithms)
Assume there are n words in this dictionary.

Algorithm 1: Check each word one by one.
In the worst case, we need n steps to find the word. We say the "time complexity" of Algorithm 1 is $O(n)$.

Algorithm 2: We know words in dictionary are in the alphabet order.

- Open a page in the middle
  (Divide the dictionary into 2 parts)
- Determine the word is before or after the page
- Then repeat this process

Using Algorithm 2, one step checks "half" $\left(\frac{1}{2}\right)$

⇒ After k steps, $\left(\frac{1}{2}\right)^k \cdot n$ words remains unchecked.

So, in the worst case, we need

$\left(\left(\frac{1}{2}\right)^k \cdot n = 1 \Rightarrow n = 2^k \Rightarrow k = \log_2 n\right)$

$\log_2 n$ steps to find the word.

⇒ The time complexity of Algorithm 2 is $O(\log_2 n)$.

Since

$$\lim_{n \to \infty} \frac{\log_2 n}{n} = 0$$

Algorithm 1 requires more steps in general

⇒ Algorithm 2 is considered to be better than Algorithm 1 #

§ Improper integrals 瑕積分

Recall that if f is continuous on $[a, b]$

then $\int_a^b f(x)\,dx$ exists. "proper integral"

If we are NOT in this case, we get improper integrals

Here we introduce two types of improper integrals

<u>Type I</u> : The integral is NOT over $[a,b]$

<u>Def</u> (§11.7)

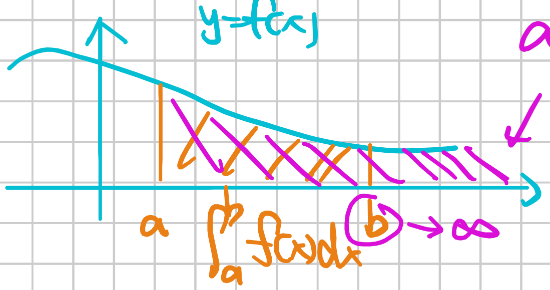
The integrals with infinite upper / lower bounds are called improper integrals (of type I)

(i) If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$\cup\!\!\mid$ $[a,b]$

area $= \int_a^\infty f(x)\,dx$



(ii) If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

(iii) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

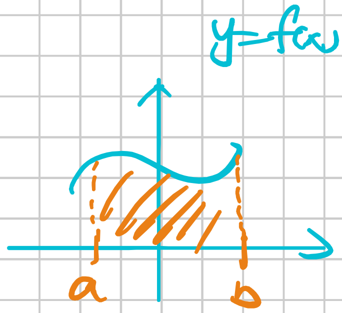
where c is any real number.

In each case, if the limit exists (and finite) then we say the improper integral converges and the limit is the value of the improper integral. If the limit does NOT exist, then we say then improper integral diverges

Recall (Fundamental Thm of Calculus)

Assume $f(x)$ is continuous and

$$F'(x) = f(x)$$



Then $$\int_a^b f(x)\,dx = F(b) - F(a) = \text{area}$$

Example

$$\int_1^{\infty} \frac{1}{x^p}\,dx = ?$$

sol

By definition, $$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$$

$= ?$

Step 1: $\int_1^b \frac{1}{x^p}\,dx = ?$

To compute it, we need to find $F(x)$ s.t.

$$F'(x) = \frac{1}{x^p} = x^{-p}$$

$\Rightarrow r = -p-1$

Recall that $(x^r)' = r \cdot x^{r-1}$ $\quad r-1 = -p$ (we have x^{-p})

$$\Rightarrow \left(x^{-p+1}\right)' = (-p+1)\, x^{-p}$$

$(r \cdot f(x))' = r \cdot f'(x)$

$$\Rightarrow \frac{1}{-p+1} \cdot \left(x^{-p+1}\right)' = \Big(\underbrace{\boxed{\frac{x^{-p+1}}{-p+1}}}_{=F(x)}\Big)' = x^{-p}$$

So

$$\int_1^b \frac{1}{x^p}\,dx = F(b) - F(1)$$

$$= \frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

Step 2: Take limit

case 1: $p \neq 1$

ans $= \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx$

$p \neq 1$

$$= \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}\right) = -\frac{1}{-p+1} = \frac{1}{p-1}$$

Note that

$$\lim_{b\to\infty} \frac{b^{-p+1}}{-p+1} = \begin{cases} \boxed{0} & -p+1<0 \iff p>1 \\ \infty \text{ (doesn't exist)} & -p+1>0 \iff p<1 \end{cases}$$

So $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } p<1 \end{cases}$

<u>Case2</u> : $p=1$ .... next time