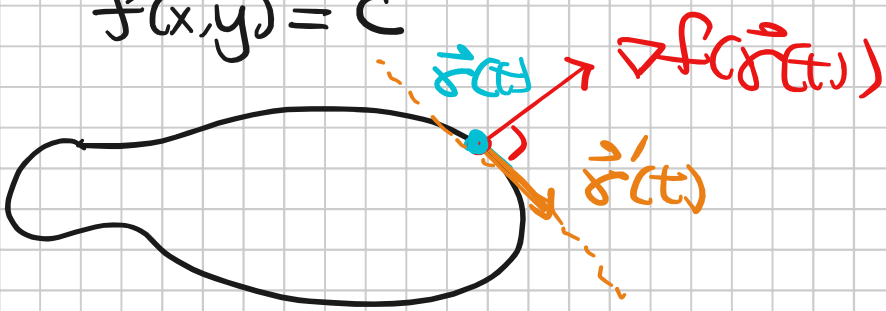
Calculus, week 11, Fall 2026

Recall

$$\left(f(\vec{\gamma}(t))\right)' = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)$$

Consider $f(x,y) = c$

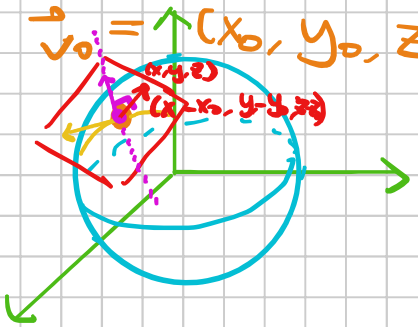


Consider a surface

$$S: f(x,y,z) = c$$

e.g. $f(x,y,z) = x^2+y^2+z^2$, $c=1$

$$f(x,y,z) = 1 = x^2+y^2+z^2$$



Assume $f(\vec{v}_0) = c$ and $\nabla f(\vec{v}_0) \neq 0$

The tangent plane of S at $\vec{v}_0$

is

$$T_{\vec{v}_0} S = \left\{ \vec{v}_0 + \vec{\gamma}'(0) \;\middle|\; \begin{array}{l} \vec{\gamma} : (-\infty, \infty) \to S \\ \vec{\gamma}(0) = \vec{v}_0 = (x_0, y_0, z_0) \end{array} \right\}$$

If $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$,

then

$$f(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = C$$

$$\Rightarrow \left(f(\gamma_1(t), \gamma_2(t), \gamma_3(t)) \right)' \Big|_{t=0} = 0$$

$\|$ chain rule

$$\nabla f(\gamma_1(0), \gamma_2(0), \gamma_3(0)) \cdot \vec{\gamma}'(0)$$

$\|$

$$\nabla f(x_0, y_0, z_0) \cdot \vec{\gamma}'(0)$$

So $T_{\vec{v}_0} S$ is the plane passing the point $\vec{v}_0$ and perpendicular to $\nabla f(x_0, y_0, z_0)$

Thm (16.4.8, 16.4.10)

Suppose $f(x_0, y_0, z_0) = C$, $\nabla f(x_0, y_0, z_0) \neq \vec{0}$

The tangent plane of

$$S: \quad f(x,y,z) = c$$

at (x_0, y_0, z_0) is given by

$$0 = \nabla f(x_0,y_0,z_0) \cdot (x-x_0,\ y-y_0,\ z-z_0)$$

$$= f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

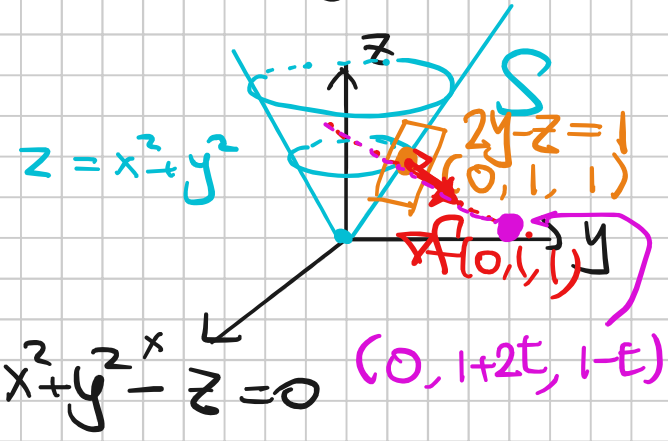
Furthermore, the <u>normal line</u> (i.e. the normal line of the tangent plane) can be parametrized by

$$\begin{cases} x = x_0 + t \cdot f_x(x_0,y_0,z_0) \\ y = y_0 + t \cdot f_y(x_0,y_0,z_0) \\ z = z_0 + t \cdot f_z(x_0,y_0,z_0) \end{cases}$$

<u>Example</u>

Consider

$$z = x^2+y^2 \iff x^2+y^2-z=0$$



Set $f(x,y,z) = x^2+y^2-z$ $\Rightarrow f_x = 2x \Rightarrow \nabla f(0,1,1) = (0,2,-1)$

$f_y = 2y$

$f_z = -1$

$S: f(x,y,z) = 0$

The tangent plane of S at $(0,1,1)$ is

$$0 = \nabla f(0,1,1) \cdot (x-0, y-1, z-1)$$

$$= (0,2,-1) \cdot (x, y-1, z-1)$$

$$= 2(y-1) - (z-1) = 2y - z - 1$$

i.e. $\boxed{2y - z = 1}$

The normal line of S at $(0,1,1)$ is

$$\begin{cases} x = 0 + t\,0 = 0 \\ y = 1 + t\cdot 2 = 1+2t \\ z = 1 + t\cdot(-1) = 1-t \end{cases}$$

Example

$f(x,y,z) = 3xy - x^3 - y^3 - z$

At what point(s) of the surface $\Leftrightarrow f(x,y,z) = 0$

$$z = 3xy - x^3 - y^3$$

is the tangent plane horizontal (i.e. the plane is given by $z = c$ for some c)

水平的

Sol

The tangent plane at $\vec{v}_0$ is horizontal

$\Leftrightarrow$ $\nabla f(\vec{v}_0) \parallel (0,0,1)$, i.e. $(f_x(\vec{v}_0), f_y(\vec{v}_0), f_z(\vec{v}_0))$

$f_x(\vec{v}_0) = 0$

$f_y(\vec{v}_0) = 0$

(where $f(x,y,z) = 3xy - x^3 - y^3 - z$)

So we shall solve the eq:

$$\begin{cases} 0 = f_x(x,y,z) = 3y - 3x^2 & (1) \\ 0 = f_y(x,y,z) = 3x - 3y^2 & (2) \end{cases}$$

(1) $\Rightarrow y = x^2$ (2)+(1) $\Rightarrow x = y^2 = x^4$

$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Rightarrow z = 0 \quad \text{or} \quad \begin{cases} x = 1 \\ y = 1 \end{cases} \Rightarrow z = 1$$

Ans: $(0,0,0)$ and $(1,1,1)$ #

Extreme values

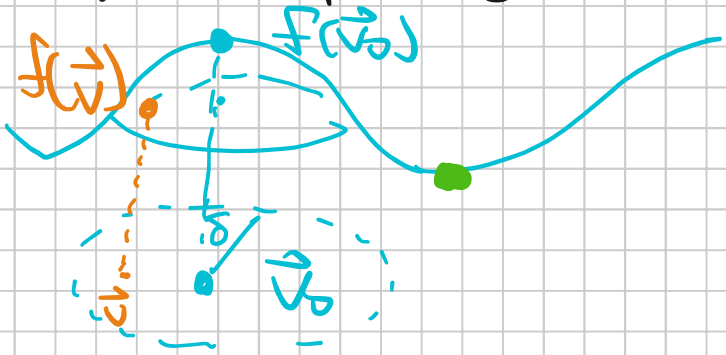
Def (Def 16.5.1)

The function $f = f(\vec{v})$, $\vec{v} = (x_1, x_2, \cdots, x_n)$

has a local maximum at $\vec{v}_0$
(local minimum)
if there exists $\delta > 0$ s.t.

$$f(\vec{v}_0) \geq f(\vec{v}) \qquad (f(\vec{v}_0) \leq f(\vec{v}))$$

when $\|\vec{v} - \vec{v}_0\| < \delta$



A local extreme value of f
= a local maximum or
a local minimum of f

Recall (*)
$f = f(x)$ has a local extreme value at $x_0 \Rightarrow f'(x_0) = 0$ or
$f'(x_0)$ does NOT exist

Thm (Thm 16.5.2)
If $f = f(\vec{v})$ has a local extreme

value at an interior point $\vec{v}_0$ of its domain, then

$$\nabla f(\vec{v}_0) = \vec{0}$$

or

$\nabla f(\vec{v}_0)$ does NOT exist.

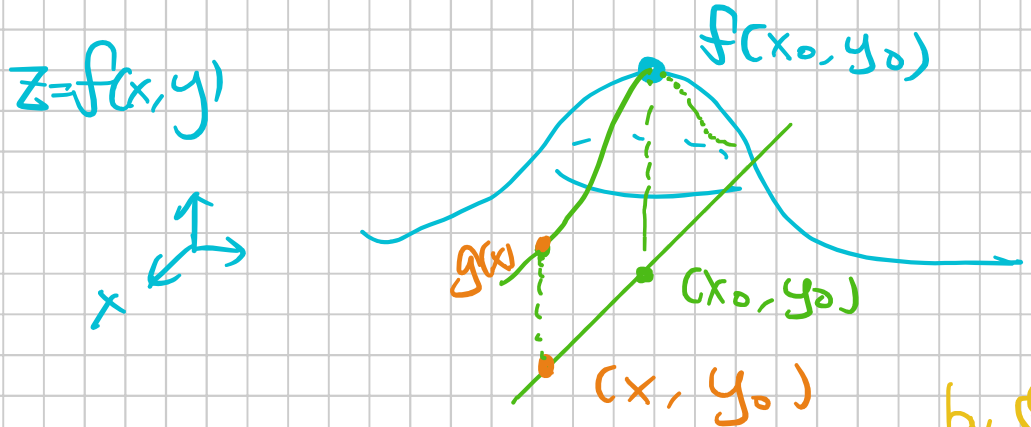
pf (case: $\vec{v} = (x, y)$ The n-variable case is similar)

Assume $\nabla f(\vec{v}_0)$ exist.

Since f has a local extreme value at $\vec{v}_0 = (x_0, y_0)$, the one-variable function

$$g(x) = f(x, y_0)$$

has a local extreme value at x_0



$$\Rightarrow \quad g'(x_0) = f_x(x_0, y_0) \overset{\text{by } \circledast}{=} 0$$

Similarly,

$$h(y) = f(x_0, y)$$

has a local extreme value at y_0

$\Rightarrow$ $h'(y_0) = f_y(x_0, y_0) = 0$

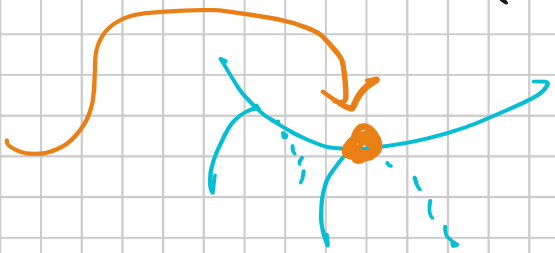
So $\nabla f(x_0, y_0) = \vec{0} = (0, 0)$. #

Def

The critical points of f are the interior points of its domain at which the gradient is zero or does not exist.

The points at which the gradient is zero are called stationary points.

The stationary points that do not give rise to local extreme values are called saddle points.



National Tsing Hua University

Calculus II –Exam 2 – Solutions

Instructor: Hsuan-Yi Liao

November 17, 2026

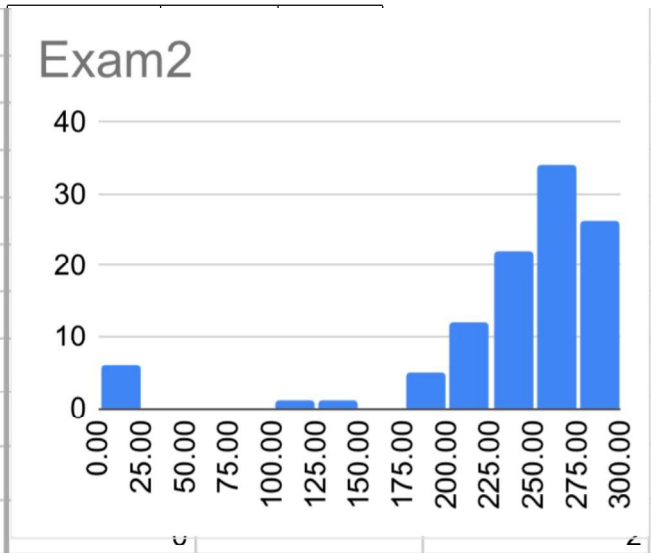
Name: _____

Student ID: _____

- This exam contains 12 pages (including this cover page) and 11 questions.
- The total number of points is 300.
- Time limit: **100 minutes** (10:10–11:50).
- Write down your computations or arguments in detail unless otherwise stated.
- The use of a calculator, cell phone, or any other electronic device is **NOT** permitted.
- The use of books or notes of any kind is **NOT** permitted.
- Keep your student ID available for inspection.

Distribution of Marks

Average	237.3394495
Average (except zero)	251.1650485
Quartile 0	0
Quartile 1	228
Quartile 2	256
Quartile 3	275
Quartile 4	300
標準差	65.71307162
非零數	103
不到60%人數	8
滿分人數	2



9	12	
10	12	
11	16	
Total:	300	

1. For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges. Explain your reasoning.

   (a) (12 points) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$.

   Solution: The sequence $1/(k \ln k)$ decreases to zero, so the alternating series test gives convergence. However,

   $$\int_2^M \frac{dx}{x \ln x} = \ln \ln M - \ln \ln 2 \longrightarrow \infty.$$

   The integral test shows that the series of absolute values diverges. Thus the series is **conditionally convergent**.

   (b) (12 points) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(\pi k/2)}{k\sqrt{k}}$.

   Solution: Since $|\sin(\pi k/2)| \le 1$,

   $$\sum_{k=1}^{\infty} \left| \frac{\sin(\pi k/2)}{k\sqrt{k}} \right| \le \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} < \infty.$$

   The comparison test and the p-series test imply **absolute convergence**.

2. Find the interval of convergence of each of the following power series.

(a) (20 points) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k2^k} x^k$.

Solution: For $x \neq 0$, the ratio of successive absolute terms tends to $|x|/2$. Thus there is absolute convergence for $|x| < 2$ and divergence for $|x| > 2$. At $x = 0$ the series is zero. At $x = -2$ it is the convergent alternating harmonic series, whereas at $x = 2$ it is the divergent harmonic series. The interval is $\boxed{[-2, 2)}$.

Alternative. The root test also gives $\sqrt[k]{|x|^k/(k2^k)} \to |x|/2$; check the same two endpoints separately.

(b) (20 points) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^2 3^k}$.

Solution: The root test gives

$$\lim_{k\to\infty} \sqrt[k]{\left|\frac{(x-1)^k}{k^2 3^k}\right|} = \frac{|x-1|}{3}.$$

There is absolute convergence for $|x - 1| < 3$ and divergence for $|x - 1| > 3$. At both endpoints $x = -2$ and $x = 4$, the series of absolute values is $\sum_{k=1}^{\infty} 1/k^2$, which converges. Hence the interval is $\boxed{[-2, 4]}$, with absolute convergence throughout.

Alternative. For $x \neq 1$, the ratio test gives the same limit; the center and endpoints are then checked directly.

3. Expand both of the following functions in powers of $x-1$, and specify the values of x for which each expansion is valid.

   (a) (30 points) $g(x) = x^{-1}$.

   Solution: Put $u = x-1$. The geometric series gives

$$\frac{1}{x} = \frac{1}{1+u} = \sum_{k=0}^{\infty}(-1)^k u^k = \boxed{\sum_{k=0}^{\infty}(-1)^k (x-1)^k}, \qquad \boxed{0 < x < 2}.$$

   At $x = 0$ or $x = 2$, the series terms fail to tend to zero; outside this interval they also fail the term test.

   Alternative. Since $g^{(k)}(1) = (-1)^k k!$, Taylor's formula gives the same coefficients. The geometric identity establishes equality to g.

   (b) (30 points) $f(x) = \sin\left(\frac{\pi x}{3}\right)$.

   Solution: Put $h = x-1$ and $a = \pi/3$. The addition formula gives

$$\sin\left(\frac{\pi x}{3}\right) = \sin(a+ah) = \frac{\sqrt{3}}{2}\cos(ah) + \frac{1}{2}\sin(ah).$$

   Substituting the cosine and sine series yields

$$\boxed{\begin{aligned}\sin\left(\frac{\pi x}{3}\right) &= \frac{\sqrt{3}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(\pi/3)^{2k}}{(2k)!}(x-1)^{2k} \\ &\quad + \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(\pi/3)^{2k+1}}{(2k+1)!}(x-1)^{2k+1}.\end{aligned}}$$

   Both series converge for every real x, so the expansion is valid for $\boxed{x \in \mathbb{R}}$.

   Alternative form. Since $f^{(n)}(1) = (\pi/3)^n \sin(\pi/3 + n\pi/2)$, the same expansion is

$$\sum_{n=0}^{\infty}\frac{(\pi/3)^n \sin(\pi/3 + n\pi/2)}{n!}(x-1)^n.$$

4. Find $f^{(9)}(0)$ for each of the following functions.

   (a) (24 points) $f(x) = x^2 \arctan x$.

   Solution: Near zero,

$$x^2 \arctan x = x^3 - \frac{x^5}{3} + \frac{x^7}{5} - \frac{x^9}{7} + \cdots .$$

   The coefficient of x^9 is $f^{(9)}(0)/9!$, so $\boxed{f^{(9)}(0) = -9!/7}$.

   Alternative. Integrate the geometric expansion $1/(1+x^2) = \sum_{k=0}^{\infty} (-1)^k x^{2k}$ from 0 to x, then multiply by x^2 to obtain the same coefficient.

   (b) (24 points) $f(x) = \dfrac{1}{1-x} + e^{2x^3}$.

   Solution: For $|x| < 1$,

$$\frac{1}{1-x} + e^{2x^3} = \sum_{n=0}^{\infty} x^n + \sum_{k=0}^{\infty} \frac{2^k}{k!} x^{3k}.$$

   The coefficient of x^9 is $1 + 2^3/3! = 7/3$. Therefore

$$\boxed{f^{(9)}(0) = \frac{7}{3}\, 9!}.$$

5. Let $\vec{a}, \vec{b} \in \mathbb{R}^3$.

(a) (16 points) Show that for all vectors $\vec{a}$ and $\vec{b}$

$$4(\vec{a} \cdot \vec{b}) = \left\|\vec{a} + \vec{b}\right\|^2 - \left\|\vec{a} - \vec{b}\right\|^2.$$

Solution: Expand the squared norms:

$$\left\|\vec{a} + \vec{b}\right\|^2 = \|\vec{a}\|^2 + 2\vec{a} \cdot \vec{b} + \left\|\vec{b}\right\|^2,$$
$$\left\|\vec{a} - \vec{b}\right\|^2 = \|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} + \left\|\vec{b}\right\|^2.$$

Subtracting gives the required identity

$$\boxed{\left\|\vec{a} + \vec{b}\right\|^2 - \left\|\vec{a} - \vec{b}\right\|^2 = 4\vec{a} \cdot \vec{b}}.$$

Alternative. In coordinates, subtract $\sum_{i=1}^3 (a_i - b_i)^2$ from $\sum_{i=1}^3 (a_i + b_i)^2$. The result is $4\sum_{i=1}^3 a_i b_i = 4\vec{a} \cdot \vec{b}$.

(b) (16 points) Show that $\vec{a} \perp \vec{b}$ iff $\left\|\vec{a} + \vec{b}\right\| = \left\|\vec{a} - \vec{b}\right\|$.

Solution: By part (a),

$$\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0 \iff \left\|\vec{a} + \vec{b}\right\|^2 = \left\|\vec{a} - \vec{b}\right\|^2 \iff \left\|\vec{a} + \vec{b}\right\| = \left\|\vec{a} - \vec{b}\right\|.$$

The last equivalence holds because norms are nonnegative.

6. (20 points) Calculate $\vec{f}'(t)$ and $\vec{f}''(t)$, where $\vec{f}(t) = e^t(\vec{\imath} - \vec{\jmath}) + e^{-2t}(\vec{\jmath} - \vec{k})$.

Solution: Differentiate the scalar coefficients of the fixed vectors:

$$\vec{f}'(t) = \boxed{e^t(\vec{\imath} - \vec{\jmath}) - 2e^{-2t}(\vec{\jmath} - \vec{k})},$$
$$\vec{f}''(t) = \boxed{e^t(\vec{\imath} - \vec{\jmath}) + 4e^{-2t}(\vec{\jmath} - \vec{k})}.$$

Equivalently, the coordinate triples are

$$\vec{f}' = (e^t, -e^t - 2e^{-2t}, 2e^{-2t}),$$
$$\vec{f}'' = (e^t, -e^t + 4e^{-2t}, -4e^{-2t}).$$

7. (20 points) Find the length of the curve. $\vec{\gamma}(t) = e^t\big(\cos t\,\vec{\imath} + \sin t\,\vec{\jmath}\big)$, from $t = 0$ to $t = \pi$.

Solution: Differentiating gives $\vec{\gamma}'(t) = e^t\big((\cos t - \sin t)\,\vec{\imath} + (\sin t + \cos t)\,\vec{\jmath}\big)$. Thus $\|\vec{\gamma}'(t)\|^2 = 2e^{2t}$ and

$$L = \int_0^\pi \|\vec{\gamma}'(t)\|\,dt = \sqrt{2}\int_0^\pi e^t\,dt = \boxed{\sqrt{2}(e^\pi - 1)}.$$

8. (16 points) Find $f_x(1,2)$, $f_y(1,2)$ and $f_{xx}(1,2)$ given that $f(x,y) = \dfrac{x}{x+y^2}$.

Solution: The quotient rule, with the other variable fixed, gives

$$f_x(x,y) = \frac{y^2}{(x+y^2)^2}, \qquad f_y(x,y) = -\frac{2xy}{(x+y^2)^2}, \qquad f_{xx}(x,y) = -\frac{2y^2}{(x+y^2)^3}.$$

At $(x,y) = (1,2)$, we obtain

$$\boxed{f_x(1,2) = \frac{4}{25}}, \qquad \boxed{f_y(1,2) = -\frac{4}{25}}, \qquad \boxed{f_{xx}(1,2) = -\frac{8}{125}}.$$

Alternative for the x derivatives. Write $f(x,y) = 1 - y^2/(x+y^2)$ and differentiate in x.

9. (12 points) Let g be a twice differentiable function of one variable and set

$$f(x, y) = g(x + y) + g(x - y).$$

Show that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}.$$

Solution: The one-variable chain rule, with the other variable fixed, gives

$$f_x = g'(x + y) + g'(x - y), \qquad f_{xx} = g''(x + y) + g''(x - y),$$
$$f_y = g'(x + y) - g'(x - y), \qquad f_{yy} = g''(x + y) + g''(x - y).$$

The second derivative in y introduces a second minus sign in the $g(x - y)$ term. Consequently $\boxed{f_{xx} = f_{yy}}$.

10. (12 points) Compute the gradient of the function $f(x, y, z) = \ln(1 + x^2 + y^2 + z^2)$.

Solution: The logarithm is defined everywhere because $1 + x^2 + y^2 + z^2 > 0$. Differentiating in each variable gives

$$f_x = \frac{2x}{1 + x^2 + y^2 + z^2}, \qquad f_y = \frac{2y}{1 + x^2 + y^2 + z^2}, \qquad f_z = \frac{2z}{1 + x^2 + y^2 + z^2}.$$

Thus

$$\boxed{\nabla f(x, y, z) = \frac{(2x, 2y, 2z)}{1 + x^2 + y^2 + z^2}}.$$

11. (16 points) Let g be a smooth function on $\mathbb{R}^2$ with $g_x(3,2) = 4$ and $g_y(3,2) = -1$. If $w(s,t) = g(s^2+t, st)$, find $w_s(1,2)$ and $w_t(1,2)$.

Solution: Write $x = s^2 + t$ and $y = st$. At $(s,t) = (1,2)$, we have $(x,y) = (3,2)$. By the chain rule,

$$w_s = 2s\, g_x(x,y) + t\, g_y(x,y), \qquad w_t = g_x(x,y) + s\, g_y(x,y).$$

Hence $\boxed{w_s(1,2) = 2(4) + 2(-1) = 6}$ and $\boxed{w_t(1,2) = 4 - 1 = 3}$.

Recall (chain rule)

$$\left(f(\vec{\gamma}(t))\right)' = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)$$

$$\left(f(\gamma_1(t), \gamma_2(t))\right)' = f_x(\gamma_1(t), \gamma_2(t)) \cdot \gamma_1'(t) + f_y(\gamma_1(t), \gamma_2(t)) \cdot \gamma_2'(t)$$

Sol

Here,

(1) $w_s(1,2)$:

$f(x,y) = g(x,y)$

$\gamma_1(s) = s^2 + t$, $\gamma_2(s) = st$

$w_s = \frac{\partial}{\partial s}(g(s^2+t, st)) = \frac{\partial}{\partial s}(f(\gamma_1(s), \gamma_2(s)))$

$= f_x(\gamma_1(s), \gamma_2(s))\, \gamma_1'(s) + f_y(\gamma_1(s), \gamma_2(s))\, \gamma_2'(s)$

$= g_x(s^2+t, st) \cdot 2s + g_y(s^2+t, st) \cdot t$

$\Rightarrow w_s(1,2) = g_x(1^2+2, 1\cdot 2) \cdot 2\cdot 1 + g_y(1^2+2, 1\cdot 2) \cdot 2$

$= g_x(3,2) \cdot 2 + g_y(3,2) \cdot 2 = 4 \cdot 2 - 2 = 6$

(4) (−1)

② $w_t(1,2)$: (s)

$\tilde{\gamma}_1(t) = s^2+t$, $\tilde{\gamma}_2 = st$

$w_t = g_x(s^2+t, st)\cdot \underbrace{\tilde{\gamma}_1'(t)}_{1} + g_y(s^2+t, st)\underbrace{\tilde{\gamma}_2'(t)}_{s}$

$w_t(1,2) = \underbrace{g_x(3,2)}_{4}\cdot 1 + \underbrace{g_y(3,2)}_{-1}\cdot 1$

$= 4 - 1 = 3$ #