Calculus, week 10, Fall 2026

Recall

If $f = f(x,y,z)$, then

$$\nabla f = (f_x, f_y, f_z)$$

Example

If $f(x,y,z) = x\sin(\pi y) + y\cos(\pi z)$,

then

$$f_x = \sin(\pi y)$$

$$f_y = x\cos(\pi y)\cdot\pi + \cos(\pi z)$$

$$f_z = -y\sin(\pi z)\cdot\pi$$

$$\Rightarrow \nabla f = (\sin(\pi y),\ \pi x\cos(\pi y) + \cos(\pi z),\ -\pi y\sin(\pi z))$$

$$\Rightarrow \nabla f(0,1,2) = (\underbrace{\sin(\pi\cdot 1)}_{0},\ \underbrace{\pi\cdot 0\cdot\cos(\pi\cdot 1)}_{0} + \underbrace{\cos(\pi\cdot 2)}_{1},\ -\pi\cdot 1\underbrace{\sin(\pi\cdot 2)}_{0})$$

$$= (0,1,0) \in \mathbb{R}^3 \quad \#$$

Thm (16.2.1)

Let f, g be smooth functions and $\alpha \in \mathbb{R}$.

(i) $\nabla(f+g) = \nabla f + \nabla g$

$= ((f+g)_x, (f+g)_y, (f+g)_z)$ $= (f_x, f_y, f_z) + (g_x, g_y, g_z)$

$= (f_x + g_x, f_y + g_y, f_z + g_z)$

(ii) $\nabla(\alpha f) = \alpha \nabla f$

(iii) $\nabla(fg) = g \cdot \nabla f + f \cdot \nabla g$

$= ((fg)_x, (fg)_y, (fg)_z)$

$= (f_x g + f g_x, f_y g + f g_y, f_z g + f g_z)$

$= (f_x g, f_y g, f_z g) + (f g_x, f g_y, f g_z)$

$= g \cdot (f_x, f_y, f_z) + f \cdot (g_x, g_y, g_z)$

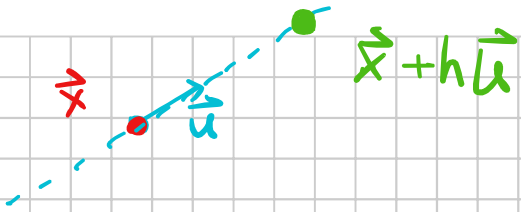
$= g(x,y,z) \cdot (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$

Directional derivative

Def (Def 16.2.2) ("direction")

A vector $\vec{u}$ is called a unit vector (單位向量)

if $\|\vec{u}\| = 1$



The limit

$$(\nabla_{\vec{u}} f)(\vec{x}) := \lim_{h \to 0} \frac{f(\vec{x}+h\vec{u}) - f(\vec{x})}{h}$$

f 沿著 $\vec{u}$ 方向的變化率

is called the directional derivative (方向導數) of f at $\vec{x}$ in the direction $\vec{u}$, provided the limit exists.

Remark

① In the textbook, there is another notation:

$$f'_{\vec{u}} = \nabla_{\vec{u}} f$$

② The assumption "$\|\vec{u}\| = 1$" is not very important. (One can compute $\lim_{h \to 0} \frac{f(\vec{x}+h\vec{u}) - f(\vec{x})}{h}$ $(= \nabla_{\vec{u}} f)$ for $\|\vec{u}\| \neq 1$).

We have "$\|\vec{u}\| = 1$" because "direction" is usually represented by a unit vector.

③ $\nabla_{\vec{u}} f(\vec{x}) = \lim_{h\to 0} \dfrac{f(\vec{x}+h\vec{u}) - f(\vec{x})}{h}$

$= \dfrac{d}{dt}\Big|_{t=0} f(\vec{x}+t\vec{u})$ "restrict f to the line $\vec{x}+t\vec{u}$" $t\in\mathbb{R}$

= rate of change of f at $\vec{x}$ in the direction $\vec{u}$

④ Partial derivatives can be considered directional derivatives:

Let $f = f(x,y,z)$.

$\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$

Then $\|\vec{i}\| = \sqrt{1^2+0^2+0^2} = 1$, $\|\vec{j}\| = 1$, $\|\vec{k}\| = 1$

and

$(\nabla_{\vec{i}} f)(\underbrace{x,y,z}_{=\vec{x}}) = \lim_{h\to 0} \dfrac{f(\vec{x}+h\vec{i}) - f(\vec{x})}{h}$

$(x,y,z) + h(1,0,0)$

$\Rightarrow (x+h, y, z)$

$= \lim_{h\to 0} \dfrac{f(x+h,y,z) - f(x,y,z)}{h}$

$= f_x(x,y,z)$

Similarly,

$$D_{\vec{j}} f(x,y,z) = f_y(x,y,z)$$

$$D_{\vec{k}} f(x,y,z) = f_z(x,y,z)$$

Thm (Thm 16.2.4)

$(u_1, u_2, u_3) = \vec{u}$

$$D_{\vec{u}} f(\vec{x}) = Df(\vec{x}) \cdot \vec{u}$$

$$= f_x(x,y,z) \cdot u_1 + f_y(x,y,z) \cdot u_2 + f_z(x,y,z) \cdot u_3$$

pf

We prove it for the case of 2 variables.

Let $\vec{u} = (u_1, u_2)$

Recall (MVT): f smooth, $\exists$ c between a and b s.t. $f(b) - f(a) = f'(c) \cdot (b-a)$

$$f(\vec{x}+h\vec{u}) - f(\vec{x})$$

$$= \underbrace{f(x_1 + hu_1, x_2 + hu_2) - f(x_1, x_2 + hu_2)}_{b \qquad\qquad a} + \underbrace{f(x_1, x_2 + hu_2) - f(x_1, x_2)}_{b \qquad\qquad a}$$

$= f_x(c_h, x_2 + hu_2) \cdot hu_1$ $\qquad$ $= f_y(x_1, c_h') \cdot hu_2$

$$\Rightarrow \frac{f(\vec{x}+h\vec{u}) - f(\vec{x})}{h} = f_x(c_h, x_2 + hu_2) \cdot u_1 + f_y(x_1, c_h') \cdot u_2$$

as $h \to 0 \longrightarrow$ $f_x(x_1, x_2) \cdot u_1 + f_y(x_1, x_2) \cdot u_2$

$= \nabla f(x_1, x_2) \cdot \vec{u}$ #

Recall

Let $f = f(\vec{x})$ be a multivariable function

$$g(t) := f(\vec{x} + t\vec{u})$$

$$\Rightarrow g'(0) = \nabla_{\vec{u}} f(\vec{x})$$

In fact, we can compute

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

$$= \lim_{h \to 0} \frac{f(\vec{x} + (t+h)\vec{u}) - f(\vec{x} + t\vec{u})}{h}$$

$$= \nabla_{\vec{u}} f(\vec{x} + t\vec{u})$$

$$\overset{\text{Thm}}{=} \nabla f(\vec{x} + t\vec{u}) \cdot \vec{u}$$

Example

Consider

$$f(x, y) = x^2 + y^2$$

Q: $D_{\vec{u}} f(1,2) = ?$

where $\vec{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$

Sol

$$D_{\vec{u}} f(1,2) \overset{\text{Thm}}{=} \nabla f(1,2) \cdot \vec{u}$$

$$f_x = 2x, \quad f_y = 2y$$

$$\Rightarrow \nabla f = (2x, 2y)$$

$$\Rightarrow \nabla f(1,2) = (2\cdot 1, 2\cdot 2)$$

$$= (2,4)$$

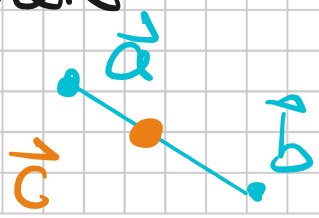
$$\Rightarrow D_{\vec{u}} f(1,2) = (2,4) \cdot \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$= 2\cdot\frac{3}{5} + 4\cdot\frac{4}{5}$$

$$= \frac{6+16}{5} = \frac{22}{5} \quad \#$$

MVT and chain rule

Thm (Thm 16.3.1) $\vec{a}, \vec{b}$ be 2 points. Let f be smooth. Then there exists $\vec{c}$ on the line



segment between $\vec{a}$ and $\vec{b}$

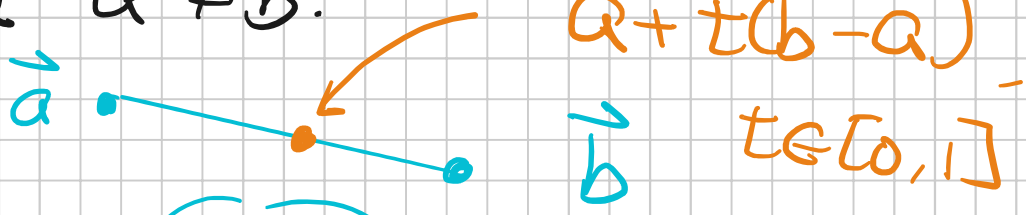
s.t.

$$\circledast \quad f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b}-\vec{a})$$

pf

If $\vec{a}=\vec{b}$, then $\circledast$ is true.

Assume $\vec{a} \neq \vec{b}$.



Let $\vec{u} = \dfrac{\vec{b}-\vec{a}}{\|\vec{b}-\vec{a}\|}$, and $\|\vec{b}-\vec{a}\|\,\vec{u} = \vec{b}-\vec{a}$

$h(t) = f(\vec{a}+t\vec{u})$

$$g(t) = f(\vec{a}+t(\vec{b}-\vec{a}))$$

$$= f(\vec{a}+t\|\vec{b}-\vec{a}\|\vec{u})$$

$$\Rightarrow g'(t) = \nabla f(\vec{a}+t\|\vec{b}-\vec{a}\|\vec{u}) \cdot \vec{u}\,\|\vec{b}-\vec{a}\|$$

$$= \nabla f(\vec{a}+t\|\vec{b}-\vec{a}\|\vec{u}) \cdot (\vec{b}-\vec{a})$$

$$h'(t) = \nabla f(\vec{a}+t\vec{u}) \cdot \vec{u}$$

$$= \nabla f(\vec{a}+t(\vec{b}-\vec{a})) \bullet (\vec{b}-\vec{a})$$

Note that $h(0) = f(\vec{a})$

$$g(0) = f(\vec{a}+0(\vec{b}-\vec{a})) = f(\vec{a})$$

$$g(1) = f(\vec{a}+(\vec{b}-\vec{a})) = f(\vec{b})$$

$h(\|\vec{b}-\vec{a}\|) = f(\vec{b})$

By MVT for $g(t)$, $\exists\ c \in (0,1)$ s.t. $c \in (0, \|\vec{b}-\vec{a}\|)$

$$\underbrace{g(1)-g(0)}_{h(\|\vec{b}-\vec{a}\|)-h(0)} = g'(c)\cdot(1-0)$$

$$\| \qquad\qquad \|$$

$$\underline{f(\vec{b})-f(\vec{a})} \qquad \underbrace{\nabla f(\underbrace{\boxed{\vec{a}+c(\vec{b}-\vec{a})}}_{\vec{c}\ \text{call it}}) \bullet (\vec{b}-\vec{a})}_{h'(c)\,\|\vec{b}-\vec{a}\|}$$

$$\Rightarrow f(\vec{b})-f(\vec{a}) = \nabla f(\vec{c}) \bullet (\vec{b}-\vec{a})$$

#

<u>Thm</u>

If $\nabla f(\vec{x}) = \vec{0} \quad \forall \vec{x} \in \mathbb{R}^3$,

then f is constant.

<u>pf</u>

$\forall \vec{x}, \vec{y} \in \mathbb{R}^3$, $\exists \vec{c}$ between $\vec{x}$ and $\vec{y}$ s.t.

$$f(\vec{x}) - f(\vec{y}) = \nabla f(\vec{c}) \bullet (\vec{x} - \vec{y})$$

$$= \vec{0}$$

$$= 0$$

$$\Rightarrow f(\vec{x}) = f(\vec{y}) \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^3$$

$\Rightarrow f$ is constant. #

Thm

If $\nabla f(\vec{x}) = \nabla g(\vec{x}) \quad \forall \vec{x} \in \mathbb{R}^3$,

then $f - g$ is constant.

Pf

$$\nabla f(\vec{x}) = \nabla g(\vec{x}) \Rightarrow \nabla(f-g)(\vec{x}) = \vec{0} \quad \forall \vec{x} \in \mathbb{R}^3$$

$\Rightarrow f - g$ is constant. #

Consider a vector-valued function $\vec{\gamma}(t)$ and the composition $f(\vec{\gamma}(t))$

Thm (Chain Rule, Thm 16.3.4)

Let $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and $f = f(x,y,z)$ be smooth

Then

$$(f \circ \vec{\gamma})'(t) = \left(f(\vec{\gamma}(t))\right)'$$

$$= \frac{d}{dt}\left(f(\gamma_1(t), \gamma_2(t), \gamma_3(t))\right)$$

$$= \nabla f(\vec{\gamma}(t)) \bullet \vec{\gamma}'(t)$$

$$= f_x(\gamma_1(t), \gamma_2(t), \gamma_3(t))\, \gamma_1'(t)$$
$$+ f_y(\gamma_1(t), \gamma_2(t), \gamma_3(t))\, \gamma_2'(t)$$
$$+ f_z(\gamma_1(t), \gamma_2(t), \gamma_3(t))\, \gamma_3'(t)$$

Idea of pf:

$$\left(f(\vec{\gamma}(t))\right)' = \lim_{h \to 0} \frac{f(\vec{\gamma}(t+h)) - f(\vec{\gamma}(t))}{h}$$

($\vec{b}$ = $\vec{\gamma}(t+h)$, $\vec{a}$ = $\vec{\gamma}(t)$)

MVT

$$= Df(\vec{C}_h) \bullet \frac{(\vec{\gamma}(t+h) - \vec{\gamma}(t))}{h}$$

($\vec{C}_h$ between $\vec{\gamma}(t)$, $\vec{\gamma}(t+h)$; $\vec{b}$ = $\vec{\gamma}(t+h)$, $\vec{a}$ = $\vec{\gamma}(t)$)

$$= \lim_{h \to 0} Df(\vec{C}_h) \bullet \frac{(\vec{\gamma}(t+h) - \vec{\gamma}(t))}{h}$$

($\vec{C}_h \to \vec{\gamma}(t)$; $\frac{(\vec{\gamma}(t+h) - \vec{\gamma}(t))}{h} \to \vec{\gamma}'(t)$)

$$= \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) \quad \#$$

Example

Find the rate of change of

$$f(x,y) = \frac{1}{3}(x^2+y^2)$$

along the curve

$$\vec{\gamma}(t) = (\cos t, 2\sin t)$$

with respect to t

sol

$$\text{Ans} = \left(f(\vec{\gamma}(t))\right)'$$

$$\overset{\text{chain rule}}{=} \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)$$

$$f_x = \frac{2}{3}x, \quad f_y = \frac{2}{3}y$$

$$\Rightarrow \nabla f(\cos t, 2\sin t)$$

$$= \left(\frac{2}{3}\cos t, \frac{2}{3}\cdot 2\sin t\right)$$

$$\vec{\gamma}'(t) = (-\sin t, 2\cos t)$$

$$\Rightarrow \text{Ans} = \left(\frac{2}{3}\cos t, \frac{4}{3}\sin t\right) \cdot (-\sin t, 2\cos t)$$

Ans $= (3\cos t, 3\sin t) \cdot (-\sin t, 2\cos t)$

$$= \frac{-2}{3}\cos t \sin t + \frac{8}{3}\sin t \cos t$$

$$= 2\cos t \sin t \quad \#$$

Recall (Chain rule)

$f = f(x,y,z)$. $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$

$$\frac{d}{dt}\Big(f(\gamma_1(t), \gamma_2(t), \gamma_3(t))\Big)$$

$$= f_x(\gamma_1(t), \gamma_2(t), \gamma_3(t))\ \gamma_1'(t)$$

$$+ f_y(\gamma_1(t), \gamma_2(t), \gamma_3(t))\ \gamma_2'(t)$$

$$+ f_z(\gamma_1(t), \gamma_2(t), \gamma_3(t))\ \gamma_3'(t)$$

$$\Updownarrow$$

$$\big(f(\vec{\gamma}(t))\big)' = \nabla f(\vec{\gamma}(t)) \bullet \vec{\gamma}'(t)$$

Example

① Assume $\underbrace{u = x^2 - y^2}_{f(x,y)}$, $x = \underbrace{t^2 - 1}_{\gamma_1(t)}$, $y = \underbrace{3\sin \pi t}_{\gamma_2(t)}$

Q: $\frac{du}{dt} = ?$

Sol

Chain rule

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

where

$$\frac{\partial u}{\partial x} = \frac{\partial (x^2-y^2)}{\partial x} = 2x = 2(t^2-1)$$

$$\frac{\partial u}{\partial y} = -2y = -6\sin(\pi t)$$

$$\frac{dx}{dt} = \frac{d(t^2-1)}{dt} = 2t$$

$$\frac{dy}{dt} = \frac{d(3\sin \pi t)}{dt} = 3\cos(\pi t)\cdot \pi$$

$z = \pi t$

chain rule: $\frac{dy}{dt} = \frac{dy}{dz}\frac{dz}{dt}$

So

$$\frac{du}{dt} = 2(t^2-1)\cdot 2t + (-6\sin(\pi t))\cdot 3\pi\cos(\pi t)$$

$$= 4t^3 - 4t - 18\pi \sin(\pi t)\cos(\pi t) \quad \#$$

② Assume

$$u = x^2 - 2xy + 2y^3$$

$$x = s^2 \ln t \qquad y = 2st^3$$

Q: $\frac{\partial u}{\partial s} = ?$ $\frac{\partial u}{\partial t} = ?$

Sol

$$\frac{\partial u}{\partial s} \overset{\text{chain rule}}{=} \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial x} = \frac{\partial(x^2 - 2xy + 2y^3)}{\partial x} = 2x - 2y$$

$$= 2s^2\ln t - 4st^3$$

$$\frac{\partial u}{\partial y} = -2x + 6y^2 = -2s^2\ln t + 24 s^2 t^6$$

$$\frac{\partial x}{\partial s} = 2s\ln t \qquad \frac{\partial x}{\partial t} = \frac{s^2}{t}$$

$$\frac{\partial y}{\partial s} = 2t^3 \qquad \frac{\partial y}{\partial t} = 6st^2$$

$$\Rightarrow \frac{\partial u}{\partial s} = (2s^2\ln t - 4st^3)\, 2s\ln t + (-2s^2\ln t + 24 s^2 t^6)\, 2t^3$$

$$\frac{\partial u}{\partial t} = (2s^2\ln t - 4st^3)\frac{s^2}{t} + (-2s^2\ln t + 24 s^2 t^6)\, 6st^2 \quad \#$$

③ (Polar coordinates)

Assume $x = r\cos\theta$, $y = r\sin\theta$

$u = u(x,y)$ is smooth

Q1: $\frac{\partial u}{\partial r} = ?$ $\frac{\partial u}{\partial \theta} = ?$

sol

By chain rule,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

($\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$)

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

($\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$)

$$= -\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta$$

#

Q2: $\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = ?$

sol

$$\text{ans} = \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(-\frac{\partial u}{\partial x} r\sin\theta + \frac{\partial u}{\partial y} r\cos\theta\right)^2$$

$$= \left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta$$

$$+ 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\cos\theta\sin\theta$$

$$+\left(\frac{\partial u}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial u}{\partial y}\right)^2\cos^2\theta$$

$$- 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\sin\theta\cos\theta$$

$$= \left(\frac{\partial u}{\partial x}\right)^2(\cos^2+\sin^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2(\sin^2\theta+\cos^2\theta)$$

$\cos^2+\sin^2\theta = 1$, $\sin^2\theta+\cos^2\theta = 1$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Δu, $\nabla^2 u$ (Laplacian) #

④ Assume

$$\underbrace{2x^2y - y^3 + 1 - x - 2y}_{u(x,y)} = 0$$

Q: $\frac{dy}{dx} = ?$

sol

$$\frac{d}{dx}(u(x,y)) = \frac{d}{dx}(0) = 0$$

|| chain rule

$$u_x \cdot \frac{dx}{dx} + u_y \cdot \frac{dy}{dx} = 0$$

$\frac{dx}{dx} = 1$

$u = 2x^2y - y^3 + 1 - x - 2y$

$$\Rightarrow \frac{dy}{dx} = \frac{-u_x}{u_y}$$

$$= -\frac{4xy - 1}{2x^2 - 3y^2 - 2} \quad \#$$

⑤ Assume $z = z(x, y)$. x, y are indep. variables

$$\underbrace{\cos(xyz) + \ln(x^2+y^2+z^2) = 0}_{u(x,y,z)}$$

Q: $\frac{\partial z}{\partial x} = ?$ $\frac{\partial z}{\partial y} = ?$

Consider y as Constant

Sol

By chain rule,

(a) $$\frac{\partial u(x,y,z)}{\partial x} = \frac{\partial 0}{\partial x} = 0$$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$= -\sin(xyz)\cdot yz + \frac{2x}{x^2+y^2+z^2}$$

$$+\left(-\sin(xyz)\cdot xz + \frac{2y}{x^2+y^2+z^2}\right)\frac{\partial y}{\partial x}$$

$$+\left(-\sin(xyz)\cdot xy + \frac{2z}{x^2+y^2+z^2}\right)\frac{\partial z}{\partial x}$$

$$= 0$$

(b) $\dfrac{\partial(u(x,y,z))}{\partial y} = \dfrac{\partial 0}{\partial y} = 0$

$$\parallel$$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}$$

$$= \left(-\sin(xyz)\cdot yz + \frac{2x}{x^2+y^2+z^2}\right)\frac{\partial x}{\partial y}^{0}$$

$$+ \left(-\sin(xyz)\cdot xz + \frac{2y}{x^2+y^2+z^2}\right)\cdot 1$$

$$+ \left(-\sin(xyz)\,xy + \frac{2z}{x^2+y^2+z^2}\right)\frac{\partial z}{\partial y} = 0$$

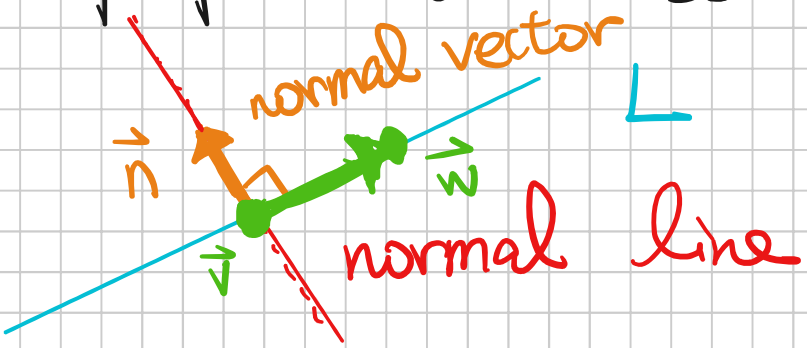
$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\left(\left(\sin(xyz)\cdot yz - \frac{2x}{x^2+y^2+z^2}\right) - \left(-\sin(xyz)\cdot xz + \frac{2y}{x^2+y^2+z^2}\right)\frac{\partial y}{\partial x}\right)}{\left(-\sin(xyz)\cdot xy + \frac{2z}{x^2+y^2+z^2}\right)}$$

$$\frac{\partial z}{\partial y} = \frac{\sin(xyz)\cdot xz - \frac{2y}{x^2+y^2+z^2}}{-\sin(xyz)\cdot xy + \frac{2z}{x^2+y^2+z^2}} \quad \#$$

Geometric meaning: gradient and normal vector 法向量

Recall that a normal vector of

a line L in $\mathbb{R}^2$ is a vector that is perpendicular to L, and the normal 法線 line of L is the line perpendicular to L



In other words, a vector $\vec{n}$ is a normal vector of L iff

$$\vec{n} \bullet (\vec{w} - \vec{v}) = 0$$

for all $\vec{v}, \vec{w} \in L$

If L is given by

$$ax + by = c$$

$\vec{v} = (v_1, v_2)$

$\vec{w} = (w_1, w_2)$

then

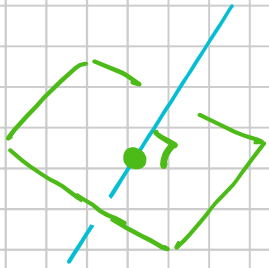
$$\begin{cases} av_1 + bv_2 = c & (1) \\ aw_1 + bw_2 = c & (2) \end{cases}$$

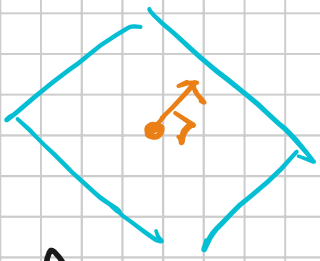
$(2)-(1)$:

$$\Rightarrow \boxed{a(w_1 - v_1) + b(w_2 - v_2)} = c - c = 0$$

$$\| \quad (a,b) \cdot (\vec{w} - \vec{v}) = 0$$

So $\vec{n} = (a,b)$ is a normal vector of L.





$P: ax+by+cz = d$

Similarly, a normal vector of a plane P in $\mathbb{R}^3$ is a vector that is perpendicular to P, and the normal line of P is the line perpendicular to P.

If P is given by

$$ax+by+cz = d,$$

then

$$\vec{n} = (a,b,c)$$

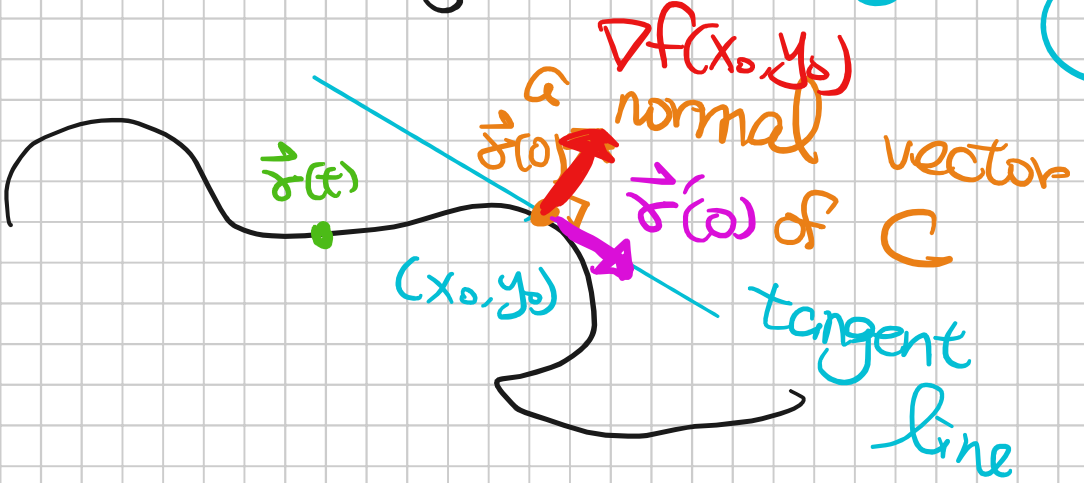
is a normal vector of P.

Consider a curve in xy-plane given by an equation

$$C: \quad f(x,y) = c$$

e.g. $x^2+y^2=1$

C: $(\cos t, \sin t)$



Suppose $(x_0, y_0) \in C$ and $\nabla f(x_0,y_0) \neq 0$.

If C is parametrized by

$$\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$$
$$\vec{\gamma}(0) = (x_0, y_0)$$

e.g. $\vec{\gamma}(t) = (\cos t, \sin t)$

$(\cos t)^2 + (\sin t)^2 = 1$

then

$$f(\gamma_1(t), \gamma_2(t)) = c$$

$\xrightarrow{\frac{d}{dt}}$

$$\frac{d}{dt}\left(f(\gamma_1(t), \gamma_2(t))\right) = \frac{dc}{dt} = 0$$

$\|$ chain rule

$$f_x(\gamma_1(t),\gamma_2(t)) \cdot \gamma_1'(t) + f_y(\gamma_1(t),\gamma_2(t))\,\gamma_2'(t)$$
$$= \nabla f(\vec{\gamma}(t)) \bullet \vec{\gamma}'(t) = 0$$

So $\nabla f(x_0, y_0) \cdot \vec{\gamma}'(0) = 0$

$\vec{\gamma}(0)$

tangent vector

a normal vector of tangent line.

Thm (16.4.4, 16.4.5)

Suppose $\nabla f(x_0, y_0) \neq 0$.

Then $\nabla f(x_0, y_0)$ is a normal vector of the tangent line of

$$f(x, y) = c$$

at (x_0, y_0)



This tangent line is given by

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$$

$$= f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0) = 0$$

equation of tangent line

Furthermore, the normal line, i.e. the normal line of the tangent line of the curve "$f(x, y) = c$" at (x_0, y_0) is given by

$$f_y(x_0, y_0) \cdot (x - x_0) - f_x(x_0, y_0) \cdot (y - y_0) = 0$$

NOTE: $(a, b) \cdot (b, -a) = ab - ba = 0$

Example

$f(x, y) = x^2 + y^2 = 1$ $\qquad (x_0, y_0) = (1, 0)$

$\nabla f(1, 0) = (2 \cdot 1, 2 \cdot 0) = (2, 0)$

