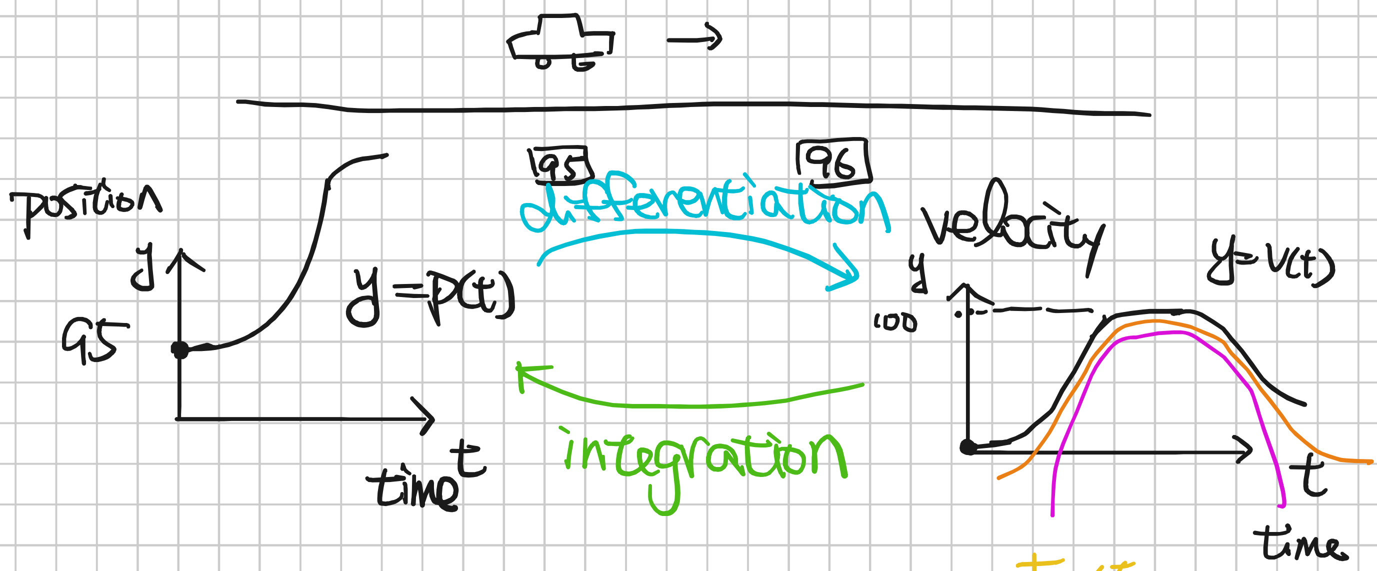
Calculus, week 1, Fall 2026

Calculus is the study of how things change. For example,



NOTE: we usually use "continuous 連續 functions 函數" to describe these data

Problem: Continuous functions can be very complicated

One possible solution:

Use some simpler functions such as polynomials 多項式 to approach continuous functions

In this process, we need "sequences 數列"

Sequences

(See Def. 11.2.1 in textbook)

A sequence (of real numbers) is usually written an $(a_n)_{n=1}^{\infty}$ where a_n are real numbers

$$(a_n)_{n=1}^{\infty} = a_1, a_2, a_3, a_3, a_5, \cdots, a_n, a_{n+1}, \cdots$$

NOT strictly increasing

For examples,

bounded above and below

① $1, 1, 1, 1, 1, \cdots$ $\xrightarrow{\times 3}$ $3, 3, 3, 3, 3, \cdots$

constant

is a sequence. ($a_n = 1$ for all $n = 1, 2, 3, \cdots$)

NOT bounded above, bounded below by 1

② $1, 2, 3, 4, \cdots$ ($a_n = n$)

is a sequence,

strictly increasing

③ $1, 5, -1, 2, \frac{1}{3}, \cdots$

is a sequence.

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences and γ is a real number.

Then we can form new sequences:

(i) $(\gamma a_n)_{n=1}^{\infty} = \gamma a_1, \gamma a_2, \gamma a_3, \ldots$

(ii) $(a_n + b_n)_{n=1}^{\infty} = a_1+b_1, a_2+b_2, a_3+b_3, \ldots$

(iii) $(a_n - b_n)_{n=1}^{\infty} = a_1-b_1, a_2-b_2, \ldots$

(iv) $(a_n \cdot b_n)_{n=1}^{\infty} = a_1b_1, a_2b_2, \ldots$

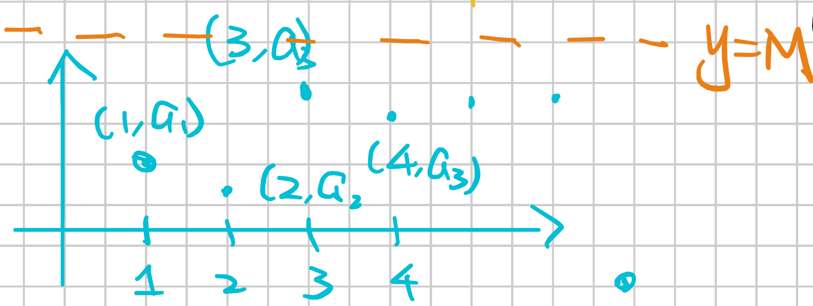
(v) $\left(\frac{1}{b_n}\right)_{n=1}^{\infty} = \frac{1}{b_1}, \frac{1}{b_2}, \ldots$

assume $b_n \neq 0$ for all $n = 1, 2, 3, \ldots$

(vi) $\left(\frac{a_n}{b_n}\right)_{n=1}^{\infty} = \frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots$

We say $(a_n)_{n=1}^{\infty}$ is ... there exists

(vii) bounded above 有上界 if $\exists$ M s.t. such that $a_n \leq M$ $\forall$ $n = 1, 2, \ldots$ for all



(viii) bounded below 有下界 if $\exists N$ s.t. $a_n \geq N$ $\forall n = 1, 2, 3, \ldots$

(ix) bounded 有界 if it is both bounded above and below

(x) strictly increasing (= increasing in textbook)

嚴格遞增

if $a_n < a_{n+1}$ $\forall n = 1, 2, \cdots$

(xi) increasing (= non decreasing in textbook)

遞增

if $a_n \leq a_{n+1}$ $\forall n = 1, 2, \cdots$

(xii) strictly decreasing (= decreasing in textbook)

嚴格遞減

if $a_n > a_{n+1}$ $\forall n = 1, 2, \cdots$

(xiii) decreasing (= non increasing in textbook)

遞減

if $a_n \geq a_{n+1}$ $\forall n = 1, 2, \cdots$

(xiv) constant if $a_n = a_{n+1}$ $\forall n = 1, 2, \cdots$

常數的

Example

① $\left(\frac{1}{n}\right)_{n=1}^{\infty} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$

is strictly decreasing (since $\frac{1}{n} \gneq \frac{1}{n+1}$ $\forall n$)

and bounded above by 1

bounded below by 0

② $((-1)^n)_{n=1}^{\infty} = \boxed{-1}, \boxed{1}, -1, 1, \ldots$

is bounded above by 1 and bounded below by -1

It is neither increasing nor decreasing

③ Using $(a_n = 1)_{n=1}^{\infty}$ and $(b_n = n)_{n=1}^{\infty}$ we can form:

- $(a_n + b_n)_{n=1}^{\infty} = (1+n)_{n=1}^{\infty}$

  $= 2, 3, 4, 5, \ldots$

  is strictly increasing and bounded below

- $\left(\dfrac{b_n}{a_n+b_n}\right)_{n=1}^{\infty} = \left(\dfrac{n}{1+n}\right)_{n=1}^{\infty}$ $= 1 - \dfrac{1}{1+n}$

  $= \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \cancel{\dfrac{1}{5}}\ \dfrac{4}{5}, \ldots$

  is strictly increasing because

$\boxed{\text{method I}}$

$$\frac{n}{1+n} < \frac{n+1}{1+n+1} = \frac{1+n}{2+n}$$

$$\Leftrightarrow \cancel{(1+n)}(2+n) \cdot \frac{n}{\cancel{1+n}} \ ? \ \frac{1+n}{\cancel{2+n}}(1+n)\cancel{(2+n)}$$

$$\| \qquad\qquad\qquad \|$$

$$n^2+2n \quad < \quad n^2+2n+1$$

$\boxed{\text{method II}}$ We also can use derivatives:

Recall:

$$f'(x)>0 \quad \forall x \Rightarrow f(x) \text{ is strictly increasing}$$

$$\Rightarrow f(x)<f(y) \quad \forall \underset{n}{x}<\underset{n+1}{y}$$

$$\Rightarrow f(n)<f(n+1)$$

Now we note that

$$\frac{b_n}{a_n+b_n}=\frac{n}{n+1}=f(n)$$

if we set $f(x)=\dfrac{x}{x+1}$

Compute $f'(x)$:

$$f'(x)=\left(\frac{\overset{f_2}{x}}{\underset{=f_1}{x+1}}\right)'$$

Recall: $\left(\dfrac{f_2}{f_1}\right)'=\dfrac{f_2'\cdot f_1-f_2\cdot f_1'}{(f_1)^2}$

$(x)'+(1)'=1+0$

$$=\frac{\overset{1}{(x)'}\cdot(x+1)-x\cdot(x+1)'}{(x+1)^2}$$

- $(x)'=1$, $(1)'=0$
- $(f_1+f_2)'=f_1'+f_2'$

$$=\frac{1\cdot(x+1)-x\cdot 1}{(x+1)^2}=\frac{1}{\underset{>0}{(x+1)^2}}>0 \quad \forall x$$

$$\Rightarrow \quad f(n) < f(n+1)$$

$$\frac{n}{n+1} = \frac{b_n}{a_n+b_n} \qquad \frac{n+1}{n+2} = \frac{b_{n+1}}{a_{n+1}+b_{n+1}}$$

$\Rightarrow \left(\frac{a_n}{a_n+b_n}\right)_{n=1}^{\infty}$ is strictly increasing #

Limits with infinity

In Calculus, we consider 3 types of limits:

- Function Type A: $\lim_{x \to c} f(x)$ ← a number

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

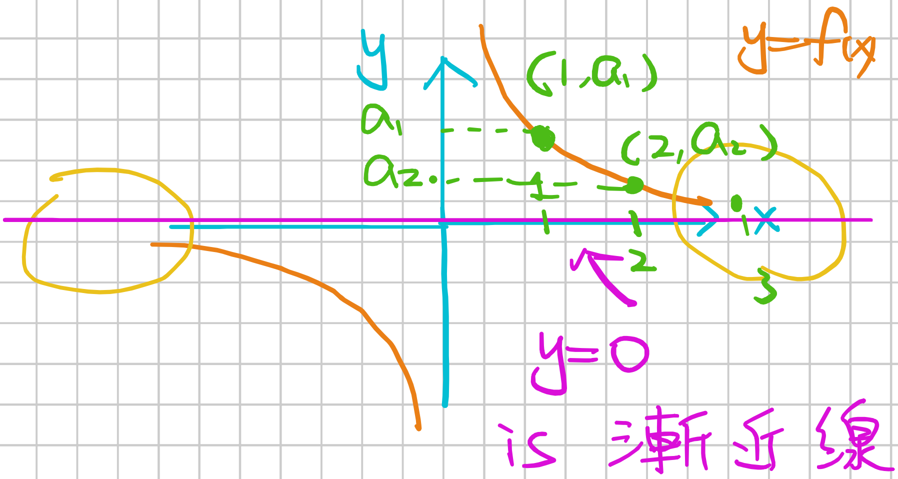
- Function Type B: $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$

Consider these 2 types now

- Sequence Type: $\lim_{n \to \infty} a_n$

Intuitive examples

① $f(x) = \frac{1}{x}$



$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

Set $a_n = f(n) = \frac{1}{n}$

$$(a_n)_{n=1}^{\infty} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

<u>Remark</u>

some number

If $\lim_{x\to\infty} f(x) = L$, and set

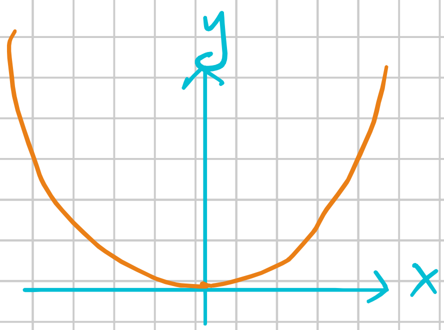
$a_n = f(n)$, $n = 1, 2, 3, \ldots$

then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(x) = L$

② Let

$$f(x) = x^2$$

" unbounded "



The limit $\lim_{x\to+\infty} f(x)$ does NOT exist

(Some people write $\lim_{x\to\infty} f(x) = +\infty$)

Similarly, the sequence

$$a_n = f(n) = n^2, \quad n = 1, 2, 3, \dots$$

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ \dots$$

diverges 發散 , $\boxed{\text{i.e.}}$ that is $\lim_{n\to\infty} a_n$

does NOT exist.

③ Let $f(x) = \cos(\pi x)$

Then the limit $\lim_{x\to\infty} f(x)$ does NOT exist

Consider

$$a_n = f(n) = \cos(n\pi) = \cancel{(-1)^{n+1}} \quad (-1)^n$$

$$\underbrace{\cos\pi}_{=-1},\ \underbrace{\cos 2\pi}_{=1},\ \underbrace{\cos 3\pi}_{=-1},\ \dots$$

The limit $\lim_{n\to\infty} a_n$ does NOT exist.

Remark

$\lim_{x\to\infty} f(x)$ does NOT exist

$\not\Rightarrow$ $\lim_{n\to\infty} f(n) = a_n$ does NOT exist

For example, $f(x) = \cos(2\pi x)$

$\lim_{x\to\infty} f(x)$ does NOT exist.

But if we set $a_n = f(n) = \cos(2n\pi) = 1$

$1, 1, 1, 1, 1, \cdots$

then $\lim_{n\to\infty} a_n$ exists, $= 1$

Q: How do we know whether a limit exist or not ?

A: method I: Use the definition of limit. This is the most fundamental method but usually very difficult.

method II: Use Convergence theorems. 定理=對的事

Thm (§11.3)

(i) The limit is unique: if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$ then $L = M$.

(ii) Every convergent sequence is bounded. 收斂的 $\lim_{n\to\infty} a_n$ exists $\Updownarrow$

(iii) Every unbounded sequence is divergent 發散的 $\Leftrightarrow$ $\lim_{n\to\infty} a_n$ does NOT exist

(iv) Every bounded above, increasing sequence is convergent

(v) Every bounded below, decreasing sequence is convergent.

Example

① $a_n = n$: $1, 2, 3, 4, \cdots$ (iii)

is divergent because it's unbounded.

② $a_n = \frac{1}{n}$: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$

is convergent because it is

by (v) decreasing and bounded below

$\frac{1}{n} > \frac{1}{n+1}$ $\qquad$ $\frac{1}{n} > 0 \ \forall n$

③ $a_n = \frac{n}{n+1}$: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$

is convergent because it is by (iv)

increasing and bounded above

$\left(\frac{n}{n+1} < \frac{n+1}{(n+1)+1} \right)$ $\qquad$ $\left(\frac{n}{n+1} < 1 \right)$

Def = Definition = 定義

A subsequence of a sequence $(a_n)_{n=1}^{\infty}$

子數列

is of the form

$$(a_{n_k})_{k=1}^{\infty} = a_{n_1}, a_{n_2}, a_{n_3}, \cdots$$

where $(n_k)_{k=1}^{\infty}$ is a strictly increasing seq. of positive integers.

Example

The seq. $(-1)^n$

$$\left(a_n = \cancel{(-1)^{n+1}}\right)_{n=1}^{\infty} = -1, 1, -1, 1, -1, 1, \ldots$$

has the following subseq:

(i) $-1, -1, -1, -1, -1, \ldots \rightarrow -1$

(ii) $1, 1, 1, 1, 1, \ldots \rightarrow 1$ *

(iii) $-1, 1, -1, 1, -1, -1, 1, -1, 1, \ldots$

NOTE: $2, 2, 2, 2, 2, \ldots$ is NOT a subseq. of $(a_n)_{n=1}^{\infty}$

Thm

Assume $(a_n)_{n=1}^{\infty}$ converges, and

$$\lim_{n\to\infty} a_n = L.$$

Then every subseq. of $(a_n)_{n=1}^{\infty}$ converges to L, as well.

Example

$a_n = \cancel{(-1)^{n+1}}$ $(-1)^n$: $-1, 1, -1, 1, \cdots$

is divergent because it has 2 subseq with different limits.

- take odd terms: $-1, -1, -1, \cdots$
  its limit $= -1$
- take even terms*: $+1, +1, +1, \cdots$
  its limit $= +1$

Thm

If $\lim_{x\to\infty} f(x) = L$, then the seq

$$a_n = f(n)$$

also converges to L.

Example

Let $f(x) = \cos(\pi x)$.

Claim: $\lim_{x\to\infty} \cos(\pi x)$ does NOT exist.

If it exists, then

$$a_n = f(n) = \cos(\pi \cdot n) = \cancel{(-1)^{n+1}} \quad (-1)^n$$

converges. (NOT true)

So $\lim_{x\to\infty} \cos(\pi x)$ does NOT exist. $\lrcorner$

Q: If a limit exists, how do we compute it?

A: Use $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to-\infty} \frac{1}{x} = 0$,

or $\lim_{n\to\infty} \frac{1}{n} = 0$

together with the operation laws of limits:

<u>Thm</u>

Assume $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist

$x\to\infty$ can be replaced by $x\to-\infty$

(i) $\lim_{x\to\infty} \left(f(x) + g(x) \right) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$

some real number

(ii) $\lim_{x\to\infty} \left(r f(x) \right) = r \cdot \lim_{x\to\infty} f(x)$

(iii) $\lim_{x\to\infty} \left(f(x) \cdot g(x) \right) = \left(\lim_{x\to\infty} f(x) \right) \cdot \left(\lim_{x\to\infty} g(x) \right)$

(iv) $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$ if $\lim_{x\to\infty} g(x) \neq 0$

Example

① $\lim\limits_{x\to\infty}\left(\frac{2}{x}+\frac{1}{x^2}\right) \overset{(i)}{=} \lim\limits_{x\to\infty}\frac{2}{x} + \lim\limits_{x\to\infty}\frac{1}{x^2}$

$\frac{1}{x}\cdot\frac{1}{x}$

(ii) // (iii)

$= 2\cdot\underbrace{\lim\limits_{x\to\infty}\frac{1}{x}}_{=0} + \underbrace{\left(\lim\limits_{x\to\infty}\frac{1}{x}\right)}_{=0}\cdot\underbrace{\left(\lim\limits_{x\to\infty}\frac{1}{x}\right)}_{=0}$

use $\lim\limits_{x\to\infty}\frac{1}{x}=0$

$= 2\cdot 0 + 0\cdot 0 = 0$ #

② $\lim\limits_{x\to\infty}\frac{(2x^2+3)\cdot\frac{1}{x^2}}{(5x^2-x)\cdot\frac{1}{x^2}}$ Idea: 利用通分讓 $\frac{1}{x}$ 出現

$= \lim\limits_{x\to\infty}\frac{\frac{2x^2}{x^2}+\frac{3}{x^2}}{\frac{5x^2}{x^2}-\frac{x}{x^2}} = \lim\limits_{x\to\infty}\frac{2+\frac{3}{x^2}}{5-\frac{1}{x}}$

NOTE: $\lim\limits_{x\to\infty} 2+\frac{3}{x^2} \overset{(i)}{=} \underbrace{\lim\limits_{x\to\infty} 2}_{=2} + \lim\limits_{x\to\infty}\frac{3}{x^2}$

indep of x

$3\cdot\frac{1}{x}\cdot\frac{1}{x}$

(ii)+(iii)

$= 2 + 3\cdot\underbrace{\left(\lim\limits_{x\to\infty}\frac{1}{x}\right)}_{=0}\cdot\underbrace{\left(\lim\limits_{x\to\infty}\frac{1}{x}\right)}_{=0}$

$= 2+3\cdot 0\cdot 0 = 2$

and

$\lim\limits_{x\to\infty}\left(5-\frac{1}{x}\right) \overset{(i)}{=} \underbrace{\lim\limits_{x\to\infty}5}_{=5} - \underbrace{\lim\limits_{x\to\infty}\frac{1}{x}}_{=0} = 5 \neq 0$

By (iv)

$$\lim_{x\to\infty} \frac{2x^2+3}{5x^2-x} = \lim_{x\to\infty} \frac{2+\frac{3}{x^2}}{5-\frac{1}{x}}$$

$$= \frac{\lim_{x\to\infty}(2+\frac{3}{x^2})}{\lim_{x\to\infty}(5-\frac{1}{x})} = \frac{2}{5}$$

#

Similarly, we have a seq. ver.:

<u>Thm</u> (Thm 11.3.7)

Assume $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, γ is a number

Then

(i) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

(ii) $\lim_{n\to\infty} (\gamma \cdot a_n) = \gamma \cdot \lim_{n\to\infty} a_n$

(iii) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$

<u>Example</u>

$\nearrow 0$

① $$\lim_{n\to\infty} \frac{(3n^4-2n^2+1)\cdot\frac{1}{n^5}}{(n^5-3n^3)\cdot\frac{1}{n^5}} = \lim_{n\to\infty} \frac{3\cdot\frac{1}{n}-2\cdot\frac{1}{n^3}+\frac{1}{n^5}}{1-\frac{3}{n^2}}$$

$\to 1$

NOTE:

$$\lim_{n\to\infty}\left(3\cdot\frac{1}{n}-2\cdot\frac{1}{n^3}+\frac{1}{n^5}\right)$$

$$\overset{(i)+(ii)}{=} 3\lim_{n\to\infty}\frac{1}{n}-2\lim_{n\to\infty}\frac{1}{n^3}+\lim_{n\to\infty}\frac{1}{n^5}$$

$$\overset{(iii)}{=} 3\cdot\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}-2\Big(\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}\Big)^3+\Big(\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}\Big)^5$$

$$= 3\cdot 0-2\cdot 0^3+0^5 = \underline{\underline{0}}$$

and

$$\lim_{n\to\infty}\left(1-\frac{3}{n^2}\right) \overset{(i)+(ii)}{=} \lim_{n\to\infty}1-3\lim_{n\to\infty}\frac{1}{n^2}$$

$$= \underbrace{\lim_{n\to\infty}1}_{=1}-3\Big(\underbrace{\lim_{n\to\infty}\frac{1}{n}}_{=0}\Big)^2 = 1-3\cdot 0$$

$$= \underline{\underline{1 \neq 0}}$$

So $\text{ans} = \frac{0}{1} = 0$ #

② $$\lim_{n\to\infty} \frac{(1-4n^7)\cdot\frac{1}{n^7}}{(n^7+12n)\cdot\frac{1}{n^7}} = \lim_{n\to\infty} \frac{\frac{1}{n^7}-4}{1+12\cdot\frac{1}{n^6}}$$

$\frac{1}{n^7} \to 0$, $\to -4$

$\frac{1}{n^6} \to 0$, $0 \neq 1$

$$= \frac{-4}{1} = -4 \#$$

③ $$\lim_{n\to\infty} \frac{(n^4-3n^2+n+2)\frac{1}{n^4}}{(n^3-7n)\frac{1}{n^4}} = \lim_{n\to\infty} \frac{1-\underbrace{\frac{3}{n^2}+\frac{1}{n^3}+\frac{2}{n^4}}_{\to 0}}{\underbrace{\frac{1}{n}-\frac{7}{n^3}}_{\to 0}}$$

does NOT exist

(If $\lim_{n\to\infty} a_n \neq 0$, $\lim_{n\to\infty} b_n = 0$, then

$\lim_{n\to\infty} \frac{a_n}{b_n}$ does NOT exist) #