

Calculus — Homework 9 (Spring 2026)

1. Simplify.

(a) $(3\vec{a} \cdot \vec{b}) - (\vec{a} \cdot 2\vec{b})$.

(c) $(\vec{a} - \vec{b}) \cdot \vec{c} + \vec{b} \cdot (\vec{c} + \vec{a})$.

(b) $\vec{a} \cdot (\vec{a} - \vec{b}) + \vec{b} \cdot (\vec{b} + \vec{a})$.

(d) $\vec{a} \cdot (\vec{a} + 2\vec{c}) + (2\vec{b} - \vec{a}) \cdot (\vec{a} + 2\vec{c}) - 2\vec{b} \cdot (\vec{a} + 2\vec{c})$.

2. Consider vectors in \mathbb{R}^3 .

(a) Find $\vec{a}, \vec{b}, \vec{c} \neq \vec{0}$ such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ but $\vec{b} \neq \vec{c}$.

(b) Show that if $\vec{u} \cdot \vec{b} = \vec{u} \cdot \vec{c}$ for all *unit vectors* \vec{u} (i.e. $\|\vec{u}\| = 1$), then $\vec{b} = \vec{c}$.

3. Let $\vec{a}, \vec{b} \in \mathbb{R}^3$.

(a) Show that for all vectors \vec{a} and \vec{b}

$$4(\vec{a} \cdot \vec{b}) = \|\vec{a} + \vec{b}\|^2 - \|\vec{a} - \vec{b}\|^2.$$

(b) Show that $\vec{a} \perp \vec{b}$ (i.e. $\vec{a} \cdot \vec{b} = 0$) iff $\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|$.

(c) Show that, if \vec{a} and \vec{b} are nonzero vectors such that

$$(\vec{a} + \vec{b}) \perp (\vec{a} - \vec{b}) \quad \text{and} \quad \|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|,$$

then the parallelogram generated by \vec{a} and \vec{b} is a square.

4. Show that

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|,$$

and the equality holds iff \vec{a} and \vec{b} are *parallel*, i.e. there exists λ such that $\vec{a} = \lambda\vec{b}$ or $\vec{b} = \lambda\vec{a}$.

5. Prove the *parallelogram law*:

$$\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 = 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2.$$

6. Calculate $\vec{f}'(t)$ and $\vec{f}''(t)$ for the following vector-valued functions in \mathbb{R}^3 .

(a) $\vec{f}(t) = (1 + 2t, 3 - t, \cos t)$.

(b) $\vec{f}(t) = e^t(1, -1, 0) + e^{-2t}(0, 1, -1)$.

(c) $\vec{f}(t) = ((t^2, -1, 0) \cdot (1, -t^2, 0))(1, 0, 0)$.

7. Find $\lim_{t \rightarrow 0} \vec{f}'(t)$ if it exists. Explain why if the limit does not exist.

(a) $\vec{f}(t) = (1 + 2t, 3 - t, \frac{t}{|t|})$.

(b) $\vec{f}(t) = e^t(1, -1, 0) + e^{-2t}(0, 1, -1)$.

(c) $\vec{f}(t) = \left(\frac{\sin t}{2t}, e^{2t}, \frac{t^2}{e^t} \right)$.

(d) $\vec{f}(t) = \left(t^2, \frac{1 - \cos t}{3t}, \frac{t}{t+1} \right)$.

8. Let \vec{f} be a differentiable vector-valued function. Show that if $\|\vec{f}(t)\| \neq 0$, then

$$\frac{d}{dt}(\|\vec{f}(t)\|) = \frac{\vec{f}(t) \cdot \vec{f}'(t)}{\|\vec{f}(t)\|}.$$

9. Show that $\|\vec{\gamma}(t)\|$ is constant iff $\vec{\gamma}(t) \cdot \vec{\gamma}'(t) = 0$ for all t .

10. Find the length of the curve $\vec{\gamma}(t)$ over the given interval.

(a) $\vec{\gamma}(t) = \left(t, \frac{2}{3}t^{3/2}, 0\right)$, from $t = 0$ to $t = 8$.

(b) $\vec{\gamma}(t) = (e^t \cos t, e^t \sin t, 0)$, from $t = 0$ to $t = \pi$.

(c) $\vec{\gamma}(t) = (t, \ln(\sec t), 3)$, from $t = 0$ to $t = \frac{\pi}{4}$.

(d) $\vec{\gamma}(t) = (t \sin t + \cos t, \sin t - t \cos t, \frac{\sqrt{3}}{2}t^2)$, from $t = 0$ to $t = 2\pi$.

11. Let

$$\vec{\gamma}(t) = (3 \cos t, 3 \sin t, 4t), \quad t \geq 0.$$

(a) Let

$$s(t) = \int_0^t \|\vec{\gamma}'(u)\| du.$$

Show that s is a one-to-one function for $t \geq 0$, and find its inverse function $\tau(s)$.

(b) Let $\vec{R}(s) = \vec{\gamma}(\tau(s))$ be the reparametrization of the curve. Show that

$$\left\| \frac{d\vec{R}}{ds} \right\| = 1.$$

(The parametrization s is called the **parametrization by arc length**.)

(c) Find the length of the curve

$$\vec{R}(s), \quad 0 \leq s \leq L.$$