Calculus — Homework 8 (Spring 2026)

1. Recall that the Bernoulli polynomial $B_n(x)$ of degree n is the unique polynomial of degree n satisfying the identity:
$$\int_x^{x+1} B_n(t)\,dt = \frac{x^n}{n!}, \qquad \forall x \in \mathbb{R}.$$
   (a) Show that $B_0(x) = 1$.
   (b) Prove that
$$B_n'(x) = B_{n-1}(x) \qquad \text{for } n \geq 1.$$
   (c) Show that $\int_0^1 B_n(t)\,dt = 0$ for all $n \geq 1$.
   (d) Using the results from parts (b) and (c), derive the explicit forms of $B_1(x)$, $B_2(x)$ and $B_3(x)$.

2. Let $B_n(x)$ denote the Bernoulli polynomial of degree k.
   (a) Prove that
$$B_n(x) = (-1)^n B_n(1-x).$$
   (b) Show that $B_n(1/2) = 0$ for all odd integers $n > 1$.

3. Using the properties of Bernoulli polynomials, prove the identity for the sum of cubes:
$$\sum_{j=1}^{n} j^3 = \Big(\sum_{j=1}^{n} j\Big)^2.$$
   (Hint: Let $S(x) = (B_2(x) - B_2(0))^2$. Differentiate $S(x)$ and relate the result to $B_3(x)$.)

4. Prove that if $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are absolutely convergent, then so is $\sum_{n=0}^{\infty} c_n$, and
$$\Big(\sum_{k=0}^{\infty} a_k\Big)\Big(\sum_{k=0}^{\infty} b_k\Big) = \sum_{n=0}^{\infty} c_n$$
   where the numbers c_n are defined by
$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

5. Evaluate the following improper integral:
$$\int_0^{\infty} e^{-10t} t^2\,dt.$$

6. Suppose $k \in \mathbb{N}$. Prove that for $x > 0$, the following improper integral is convergent and satisfies the identity:
$$\int_0^{\infty} t^{x-1} \cdot \frac{e^{-t} - e^{-(k+1)t}}{1 - e^{-t}}\,dt = \Big(1 + \frac{1}{2^x} + \cdots + \frac{1}{k^x}\Big)\Gamma(x).$$

7. Recall that for $x, y > 0$, the Beta function $B(x, y)$ is:
$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$
   Prove the following properties:
   (a) For $p, q > 0$, show that $B(p,q) = B(p+1,q) + B(p,q+1)$.
   (b) For $p, q > 0$, show that $B(p,q+1) = \frac{q}{p}B(p+1,q)$.
   (c) For $p, q > 0$, show that $B(p,q+1) = \frac{q}{p+q}B(p,q)$.

(d) For $p > 0$ and $n \in \mathbb{N}$, show that $B(p, n+1) = \dfrac{n!}{p(p+1)\cdots(p+n)}$.

8. For $p, q > 0$, prove the following equality:

$$\int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}\, dx = 2^{p+q-1}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

9. Evaluate the following integrals:

   (a) $\displaystyle\int_0^{\pi/2} (\sin 2\theta)^{100}\, d\theta$.

   (b) $\displaystyle\int_0^{\infty} \frac{u^{10}}{(1+u)^{1000}}\, du$.

   (c) $\displaystyle\int_2^{3} (t-2)^5(t-3)^4\, dt$.

   (d) $\displaystyle\int_{-a}^{a} (a+x)^p(a-x)^q\, dx$, where $a, p, q > 0$.

10. Let $v_n(r)$ denote the n-dimensional volume of the ball $\left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j^2 \le r^2\right\}$. Note that we have the following recursive relation:

$$v_n(r) = \int_{-r}^{r} v_{n-1}(\sqrt{r^2 - x^2})\, dx,$$

where $v_1(r) = 2r$, $v_2(r) = \pi r^2$, and $v_3(r) = \dfrac{4\pi r^3}{3}$. Prove by induction that:

$$v_n(r) = \frac{\pi^{n/2} r^n}{\Gamma(\frac{n}{2}+1)}.$$

(Hint: Use the substitution $x = r\cos\theta$ and the Beta function identity $B(x, y) = \displaystyle\int_0^{\pi/2} 2\sin^{2x-1}\theta\cos^{2y-1}\theta\, d\theta$.)

11. Recall that the Legendre polynomial $P_n(x)$ of degree n is defined by the formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

Use the properties of the Gamma function and Beta function to compute the value of the integral $\displaystyle\int_{-1}^{1} (P_n(x))^2\, dx$.

12. Let $a < b$ be real numbers. Recall that there exists a unique polynomial $L_n(x)$ of degree n that satisfies the following two conditions:

   (i) $\displaystyle\int_a^b L_n(x)x^j\, dx = 0$ for all $j = 0, 1, \ldots, n-1$;

   (ii) $L_n(b) = 1$.

   We consider the following properties of $L_n(x)$:

   (a) Prove that $L_n(x) = P_n\left(\dfrac{2x - (a+b)}{b-a}\right)$, where P_n is the Legendre polynomial of degree n.

   (b) Evaluate the integral:

$$\int_a^b (L_n(x))^2\, dx.$$