Calculus — Homework 7 (Spring 2026)

1. Does there exists a 2π-periodic continuous function whose Fourier series is $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sin kx$? Justify your answer.

2. Let f be a real-valued continuous function on the interval $[-\pi, \pi]$, and define the function $F \colon \mathbb{R} \to \mathbb{R}$ by

$$F(a) = \int_{-\pi}^{\pi} \left(f(x) - \frac{a}{2} \right)^2 dx.$$

   Suppose the following conditions are satisfied:

   (i) $f(x) \leq 0$ for all $x \in [-\pi, \pi]$,

   (ii) $\int_{-\pi}^{\pi} f(x)^2 \, dx = 10\pi$,

   (iii) The minimum value of $F(a)$ is 5π.

   Determine the value of the integral $\int_{-\pi}^{\pi} f(x) \, dx$.

3. Recall that for a piecewise continuous 2π-periodic function $f(x)$ with Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)],$$

   the Parseval identity is given by:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

   (a) Find the Fourier series of the 2π-periodic function $f(x)$ defined by $f(x) = x^2$ for $x \in (-\pi, \pi]$.

   (b) By applying the Parseval identity to the function $f(x)$ and the series found in part (a), prove that:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

4. Prove that there exists a unique polynomial $P_n(x)$ of degree n that satisfies the following conditions:

   (i) $\int_{-1}^{1} P_n(x) x^j \, dx = 0$ for $j = 0, 1, \cdots, n-1$;

   (ii) $P_n(1) = 1$.

   Furthermore, show that such a polynomial is the Legendre polynomial of degree n, given by the formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

5. Let $P_n(x)$ be the Legendre polynomial of degree n as before. Prove that $y(x) = P_n(x)$ is a solution to the following Legendre differential equation:

$$\frac{d}{dx} \left((1 - x^2) \frac{dy}{dx} \right) + n(n+1) y = 0. \tag{1}$$

   (Hint: Use the previous exercise.)

6. Let $a < b$ be real numbers.

   (a) Prove that there exists a unique polynomial $L_n(x)$ of degree n that satisfies the following two conditions:

      (i) $\int_{a}^{b} L_n(x) x^j \, dx = 0$ for all $j = 0, 1, \ldots, n-1$;

(ii) $L_n(b) = 1$.

(b) Determine the second-order differential equation, analogous to the Legendre differential equation (1), for which $L_n(x)$ is a solution.

7. Let $P_n(x)$ denote the Legendre polynomial of degree n. Evaluate the following integral:

$$\int_{-1}^{1} \Big(P_2(x) + 2P_3(x) - P_5(x)\Big)\Big(P_3(x) + P_4(x) + P_5(x)\Big)\, dx.$$

8. Let $P_n(x)$ denote the Legendre polynomial of degree n. Given a continuous function f on $[-1, 1]$, define

$$c_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x)\, dx, \quad k = 0, 1, \ldots, N.$$

Prove that for any polynomial $Q(x)$ of degree at most N,

$$\int_{-1}^{1} \left| f(x) - \sum_{k=0}^{N} c_k P_k(x) \right|^2 dx \leq \int_{-1}^{1} |f(x) - Q(x)|^2\, dx.$$

Furthermore, show that the equality holds if and only if $Q(x) = \sum_{k=0}^{N} c_k P_k(x)$.