Calculus — Homework 6 (Spring 2026)

1. (Riemann–Lebesgue Lemma) If $f(x)$ is a piecewise continuous function on $[a, b]$, then

$$\lim_{\lambda\to\infty} \int_a^b f(x) \sin \lambda x \, dx = 0 \qquad \text{and} \qquad \lim_{\lambda\to\infty} \int_a^b f(x) \cos \lambda x \, dx = 0,$$

   where $\lambda \in \mathbb{R}$. (See the proof of Lemma 9.32 in the lecture notes.)

2. Suppose a is not an integer. Define $f(x) = \cos ax$ and $g(x) = \sin ax$, where $x \in (-\pi, \pi]$. Extend these functions to periodic functions with period 2π, and find Fourier series of them.

3. On the interval $[-\pi, \pi]$, recall that if a function is even, then its Fourier sine coefficients b_n vanish; if a function is odd, its Fourier cosine coefficients a_n vanish.

   Suppose a function $f(x)$ is defined on $[0, \pi]$. We define its **even extension** $\tilde{f}_e$ and **odd extension** $\tilde{f}_o$ on $[-\pi, \pi]$ as follows:

$$\tilde{f}_e(x) = f(|x|),$$
$$\tilde{f}_o(x) = \begin{cases} f(x), & \text{if } x \in (0, \pi] \\ 0, & \text{if } x = 0 \\ -f(-x), & \text{if } x \in [-\pi, 0). \end{cases}$$

   The **Fourier cosine series** of $f(x)$ is defined as the Fourier series of (the 2π-periodic extension of) $\tilde{f}_e(x)$, and the **Fourier sine series** of $f(x)$ is the Fourier series of (the 2π-periodic extension of) $\tilde{f}_o(x)$.

   Let $a \notin \mathbb{Z}$ and $0 < c < \pi$. Consider the following functions defined on $[0, \pi]$:

$$f(x) = \sin ax,$$
$$g(x) = \begin{cases} 1, & \text{if } 0 \le x \le c \\ 0, & \text{if } c < x \le \pi. \end{cases}$$

   Find the Fourier sine series of $g(x)$ and the Fourier cosine series of $f(x)$.

4. Let $f(x)$ be the periodic function with period 2π such that $f(x) = |x|$ for $x \in (-\pi, \pi]$. Find the Fourier series of $f(x)$.

5. Let $g(x) = \cos x$, for $x \in [0, \pi]$. Find the Fourier sine series of $g(x)$.

6. Let $f(x) = \cos^6 x$. Define

$$F(c_0, c_1, c_2, d_1, d_2) = \int_{-\pi}^{\pi} \left| f(x) - \frac{c_0}{2} - c_1 \cos x - d_1 \sin x - c_2 \cos 2x - d_2 \sin 2x \right|^2 dx.$$

   Find the minimum value of F.

7. Suppose $f(x)$ is a periodic piecewise continuous function with period 2π, and the Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

   Define

$$E_n = \int_{-\pi}^{\pi} \left| f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right|^2 dx.$$

   (a) Prove that the sequence $(E_n)_{n=1}^{\infty}$ is a decreasing sequence.

   (b) Suppose there exists $n_0 \in \mathbb{N}$ such that $E_{n_0} = 0$. Prove that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{a_0^2}{2} + \sum_{k=1}^{n_0} (a_k^2 + b_k^2).$$

8. Show that the Fourier series of the function $f(x) = x^4$ on the interval $[-\pi, \pi]$ is given by:

$$x^4 \sim \frac{\pi^4}{5} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{8\pi^2}{k^2} - \frac{48}{k^4} \right) \cos(kx).$$

By evaluating this series at a specific point $x \in [-\pi, \pi]$ and utilizing the known result $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, prove that:

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$