Calculus — Homework 5 (Spring 2026)

1. Expand $f(x)$ in powers of x.

   (a) $f(x) = \dfrac{1}{(1-x)^2}$.

   (b) $f(x) = \ln(2-3x)$.

   (c) $f(x) = \sin(x^2)$.

   (d) $f(x) = x^2 \arctan x$.

   (e) $f(x) = \dfrac{1}{1-x} + e^{2x^3}$.

   (f) $f(x) = \cosh x \, \sinh x$.

2. Find $f^{(9)}(0)$.

   (a) $f(x) = \ln(2-3x)$.

   (b) $f(x) = x\sin(x^2)$.

   (c) $f(x) = x^2 \arctan x$.

   (d) $f(x) = \dfrac{1}{1-x} + e^{2x^3}$.

3. Set $f(x) = \dfrac{e^x - 1}{x}$.

   (a) Expand $f(x)$ in a power series.

   (b) Differentiate the series and show that

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$$

4. Let α be an arbitrary real number (not necessarily an integer). Set

$$f(x) = (1+x)^{\alpha}.$$

   (a) Show that the Taylor series of $f(x)$ can be written as

$$1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

   which is called the **binomial series**.

   (b) Show that the binomial series converges absolutely on $(-1,1)$.

   (c) Let

$$\varphi(x) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k, \qquad \forall x \in (-1,1).$$

   Use term-by-term differentiation to show that

$$\begin{aligned} (1+x)\varphi'(x) &= \alpha\varphi(x), \qquad \forall x \in (-1,1), \\ \varphi(0) &= 1. \end{aligned}$$

   (d) Prove that

$$f(x) = \varphi(x), \qquad \forall x \in (-1,1).$$

   (Hint: Differentiate $\dfrac{\varphi(x)}{(1+x)^{\alpha}}$.)

   That is,

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

   for all $x \in (-1,1)$.

   In particular, we have

$$\begin{aligned} (1+x)^2 &= 1 + 2x + x^2, \\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3, \end{aligned}$$

   and

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

   if n is a positive integer.

5. For non-negative integers m and n, prove that

   (a) $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m = n = 0 \end{cases}$

   (b) $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$ for all m, n.

6. Find the Fourier series of $f(x)$, where $f(x)$ is the periodic function with period 2π satisfying the following condition:

   (a) $f(x) = x^2, \quad \forall x \in (0, 2\pi];$

   (b) $f(x) = x^2, \quad \forall x \in (-\pi, \pi];$

   (c) $f(x) = \begin{cases} 0, & \text{if } -\pi < x \leq 0, \\ x, & \text{if } 0 \leq x \leq \pi. \end{cases}$

7. Recall the identities $\sin 3x = 3\sin x - 4\sin^3 x$ and $\cos 3x = 4\cos^3 x - 3\cos x$. Use these to answer the following questions.

   (a) Find the Fourier series of $\cos^3 x$ and $\sin^5 x$.

   (b) $\int_{-\pi}^{\pi} \sin^5 x \cdot \sin 5x\,dx = ?$

   (c) $\int_{-\pi}^{\pi} \sin^5 x \cdot \cos^3 x\,dx = ?$

   (d) $\int_{-\pi}^{\pi} |\sin^5 x + \cos^3 x|^2\,dx = ?$

   (e) Use induction to find the value of $\int_{-\pi}^{\pi} \sin^{2n+1} x \cdot \sin(2n+1)x\,dx$.