Calculus — Homework 4 (Spring 2026)

1. Expand $g(x)$ in the powers of $x - 1$ and specify the values of x for which the expansion is valid.

   (a) $g(x) = e^{-4x}$.

   (b) $g(x) = \sin \pi x$.

   (c) $g(x) = \cos(\frac{1}{2}\pi x)$.

2. Find the Taylor expansion of the following functions at $x = a$:

   (a) $\cos^2 x$, $\quad a = 0$.

   (b) $\sin x$, $\quad a = \dfrac{\pi}{4}$.

   (c) $x \sin x$, $\quad a = \dfrac{\pi}{2}$.

   (d) $\sinh x$, $\quad a = 1$.

   (e) $\displaystyle\int_0^x \frac{\sin t}{t}\, dt$, $\quad a = 0$.

   (f) $\displaystyle\int_0^x \frac{\arctan t}{t}\, dt$, $\quad a = 0$.

3. Use the Taylor expansion of $f(x) = x \sin x$ to find the value of $f^{(100)}(\frac{\pi}{2})$.

4. Suppose that the series $\displaystyle\sum_{k=0}^{\infty} a_k(-3)^k$ converges. What can you conclude about the convergence of the following series?

   (a) $\displaystyle\sum_{k=1}^{\infty} a_k 2^k$.

   (b) $\displaystyle\sum_{k=2}^{\infty} a_k 3^k$.

   (c) $\displaystyle\sum_{k=3}^{\infty} (-1)^k |a_k|$.

   (d) $\displaystyle\sum_{k=4}^{\infty} a_k 4^k$.

5. Find the interval of convergence.

   (a) $\displaystyle\sum_{k=1}^{\infty} k x^k$.

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k)!} x^{4k}$.

   (c) $\displaystyle\sum_{k=1}^{\infty} (-k)^{2k} x^{2k}$.

   (d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k 2^k} x^k$.

   (e) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k} (x+1)^k$.

   (f) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{k!}{k^3} (x-1)^k$.

   (g) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{2^k} (x-2)^k$.

   (h) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(\ln k)^k} (x-1)^k$.

   (i) $\displaystyle\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^k (x+2)^k$.

6. Sum the series.

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} x^{3k+1}$.

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{3k}{k!} x^{3k-1}$.

   (c) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k(k+1)}$.

   (d) $\displaystyle\sum_{k=1}^{\infty} k^2 x^k$.

   (e) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)2^{2k+1}}$.

   (f) $\displaystyle\sum_{k=1}^{\infty} \frac{k(k+1)}{3^k}$.

7. Suppose that the power series $\displaystyle\sum_{k=0}^{\infty} a_k x^k$ and $\displaystyle\sum_{k=0}^{\infty} b_k x^k$ converge to the same value for all $x \in (-r, r)$ with $r > 0$.
   Show that $a_k = b_k$ for all $k \geq 0$.