Calculus — Homework 3 (Spring 2026)

1. Does the series converge absolutely, converge conditionally, or diverge? Explain your reasoning. (Here, $[x]$ denotes the greatest integer less than or equal to x.)

   (a) $\displaystyle\sum_{k=1}^{\infty}(-1)^k.$

   (b) $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}.$

   (c) $\displaystyle\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k}.$

   (d) $\displaystyle\sum_{k=2}^{\infty}(-1)^k\frac{k}{\ln k}.$

   (e) $\displaystyle\sum_{k=1}^{\infty}(-1)^k k\sin(1/k).$

   (f) $\displaystyle\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{k\sqrt{k}}.$

   (g) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}\sin(\frac{k\pi}{2}).$

   (h) $\displaystyle\sum_{k=2}^{\infty}(-1)^k\sin(\frac{\pi}{k})\tan(\frac{\pi}{2^k}).$

   (i) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{[k/3]}}{k}.$

2. Let f be a function which can be differentiated infinitely many times on $(-1,1)$. The **n-th Taylor polynomial** of $f(x)$ is
$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$
The **n-th remainder** of $f(x)$ is
$$R_n(x) = f(x) - P_n(x).$$

   (a) Prove that
$$R_2(x) = \frac{1}{2}\int_0^x f^{(3)}(t)\cdot(x-t)^2dt$$
for each $x \in (-1,1)$.

   (b) Find P_4 for $f(x) = \sqrt{1+x}$.

   (c) Show that if $f(x) = \sqrt{1+x}$, then
$$|R_2(x)| < \frac{\sqrt{2}}{32}, \qquad \forall\, x \in (-1/2, 1/2).$$

3. Prove that, for $x \in (0,1)$,
$$\ln(1+x) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}x^k.$$

4. Expand $g(x)$ in the powers of $x-1$ and specify the values of x for which the expansion is valid.

   (a) $g(x) = 3x^3 - 2x^2 + 4x + 1.$

   (b) $g(x) = x^{-1}.$

   (c) $g(x) = \ln(1+2x).$

5. Let $v(x)$ be a non-negative continuous function on $[a,b]$ with $a < b$. Suppose there exists some $x_0 \in [a,b]$ such that $v(x_0) > 0$. Prove that
$$\int_a^b v(x)\,dx > 0.$$