Calculus — Homework 2 (Spring 2026)

1. Determine whether the series converges or diverges. If it converges, find the sum of the series. If it diverges, explain why.

   (a) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^2 - k}$.

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{5^k}$.

   (c) $\displaystyle\sum_{k=0}^{\infty} \frac{3^{k-1}}{4^{3k+1}}$.

   (d) $\displaystyle\sum_{k=1}^{\infty} \Big(\frac{k-1}{k}\Big)^k$.

2. Determine whether the series converges or diverges. Explain why.

   (a) $\displaystyle\sum_{k=2}^{\infty} \frac{k}{k^3 - k}$.

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$.

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$.

   (d) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1 + 2\ln k}$.

   (e) $\displaystyle\sum_{k=1}^{\infty} k^2 2^{-k^3}$.

   (f) $\displaystyle\sum_{k=1}^{\infty} \frac{2 + \cos k}{\sqrt{k+1}}$.

   (g) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{k!}$.

   (h) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k 2^k}$.

   (i) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{e^k}$.

   (j) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{3^{k^2}}$.

   (k) $\displaystyle\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdots 2k}{(2k)!}$.

   (l) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^{k/2}}$.

3. Prove that the series $\displaystyle\sum_{k=1}^{\infty} \sin\left(\frac{\pi}{2^k}\right)$ is convergent.

4. Suppose that $a_n > 0$ for all $n \in \mathbb{N}$ and that $\displaystyle\lim_{n\to\infty} n a_n = c$, where $c > 0$.

   (a) Prove that $\lim_{n\to\infty} a_n = 0$.

   (b) Prove that the series $\displaystyle\sum_{k=1}^{\infty} k a_k$ is divergent.

5. Suppose that the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ is convergent. Prove that the series $\displaystyle\sum_{k=1}^{\infty} a_k^2$ is convergent.

6. Suppose that the series $\displaystyle\sum_{k=1}^{\infty} a_k^2$ is convergent, where $a_k \geq 0$.

   (a) Is $\displaystyle\sum_{k=1}^{\infty} a_k$ necessarily convergent?

   (b) Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{a_k}{k}$ is convergent.

7. Let $(a_k)_{k=1}^{\infty}$ be a decreasing sequence of positive real numbers.

   (a) Prove that the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges if and only if the series $\displaystyle\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. (This statement is known as the Cauchy Condensation Test. Hint: See Problem 9 on Homework 2 from Calculus (I), Fall 2025.)

   (b) Use the result from part (a) to prove that the p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges for $p \leq 1$ and converges for $p > 1$.

   (c) Is the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k \log k}$ convergent?

(d) Prove or disprove the following statement: The series $\sum_{k=1}^{\infty} a_k$ converges if and only if the condensation series $\sum_{k=0}^{\infty} 3^k a_{3^k}$ converges.

(e) Let $(b_k)_{k=1}^{\infty}$ be a decreasing sequence of real numbers such that $\lim_{k\to\infty} b_k = 0$. Suppose that $\sum_{k=1}^{\infty} b_{3^k}$ is convergent. Prove that the series $\sum_{k=1}^{\infty} \frac{b_k^2}{k^2}$ is also convergent.

8. Let $(a_k)_{k=1}^{\infty}$ be a decreasing sequence of positive real numbers. Suppose that $\sum_{k=1}^{\infty} a_k$ is convergent. Prove that $\lim_{n\to\infty} n a_n = 0$.

9. Let $(a_k)_{k=1}^{\infty}$ be a sequence of positive real numbers. Prove that

$$\varliminf_{k\to\infty} \frac{a_{k+1}}{a_k} \le \varliminf_{k\to\infty} (a_k)^{1/k} \le \varlimsup_{k\to\infty} (a_k)^{1/k} \le \varlimsup_{k\to\infty} \frac{a_{k+1}}{a_k}.$$

10. Compute the limit $\lim_{n\to\infty} \sqrt[n]{\frac{(n!)^2}{(2n)!}}$.