Calculus — Homework 1 (Spring 2026)

1. Assume $|r| < 1$. Prove that $\lim_{n\to\infty} r^n = 0$.

2. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences. Suppose that $\lim_{n\to\infty} b_n \neq 0$.
   (a) Prove that there exists M such that $b_n \neq 0$ for any $n \geq M$.
   (b) Prove that $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

3. State whether the sequence converges and, if it does, find the limit.
   (a) $a_n = 2^n + 2^{2/n}$.
   (b) $a_n = \left(\frac{1}{n}\right)^{n+1}$.
   (c) $a_n = \frac{\ln(n+1)}{n}$.
   (d) $a_n = \frac{3^{100n}}{2n!}$.
   (e) $a_n = \int_{-n}^{0} e^{2x}\, dx$.
   (f) $a_n = n^2 \sin\frac{\pi}{n}$.
   (g) $a_n = \left(\frac{n-1}{n}\right)^n$.
   (h) $a_n = \int_0^{1/n} \cos e^x\, dx$.
   (i) $a_n = \left(\frac{1}{2} + \frac{1}{n}\right)^{3n}$.

4. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers, and $\alpha = \overline{\lim_{n\to\infty}}\, a_n$. Recall that

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} \left(\sup\{a_n, a_{n+1}, \cdots\} \right).$$

   (a) Prove that for any $\epsilon > 0$, there exists $N = N_\epsilon$ such that $a_n < \alpha + \epsilon$ for any $n \geq N$.
   (b) Prove that for any $\epsilon > 0$, there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ such that $a_{n_k} > \alpha - \epsilon$ for any $k \in \mathbb{N}$.

5. In this problem, we consider the following question: Suppose we know that a sequence $(a_n)_{n=1}^{\infty}$ satisfies the condition that for any $\epsilon > 0$, there exists $N = N_\epsilon \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$ for all $n \geq N$. Can we conclude that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence?
   (a) Let $a_n = \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}}$. Prove that for any $\epsilon > 0$, there exists $N = N_\epsilon \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$ for all $n \geq N$.
   (b) Prove that the above sequence $(a_n)_{n=1}^{\infty}$ is not a Cauchy sequence.
   (c) Let $(b_n)_{n=1}^{\infty}$ be a sequence satisfying the following condition:

$$|b_n - b_{n+1}| \leq \frac{1}{n(n+1)}, \qquad \forall n \in \mathbb{N}.$$

   Prove that $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

6. Let j be a positive integer.
   (a) Show that

$$\sum_{k=0}^{\infty} a_k \quad \text{converges} \qquad \text{iff} \qquad \sum_{k=j}^{\infty} a_k \quad \text{converges.}$$

   (b) Show that if $\sum_{k=0}^{\infty} a_k = L$, then $\sum_{k=j}^{\infty} a_k = L - \sum_{k=0}^{j-1} a_k$.
   (c) Show that if $\sum_{k=j}^{\infty} a_k = M$, then $\sum_{k=0}^{\infty} a_k = M + \sum_{k=0}^{j-1} a_k$.

7. Let $\alpha, \beta, \gamma \in \mathbb{R}$ satisfy the condition that $\alpha + \beta + \gamma = 0$. Define

$$c_n = \begin{cases} \alpha, & \text{if } n \equiv 1 \mod 3; \\ \beta, & \text{if } n \equiv 2 \mod 3; \\ \gamma, & \text{if } n \equiv 0 \mod 3. \end{cases}$$

   Denote

$$s_n = \sum_{k=1}^{n} c_k \qquad \text{and} \qquad \sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n).$$

(a) Find a general formula for the value of s_n.

(b) Prove that for $x \in (0, 1)$,

$$\sum_{n=1}^{\infty} c_n x^n = \frac{\alpha x + \beta x^2 + \gamma x^3}{1 - x^3}.$$

(c) Prove that $(\sigma_n)_{n=1}^{\infty}$ is convergent, and

$$\lim_{n \to \infty} \sigma_n = \lim_{x \to 1^-} \frac{\alpha x + \beta x^2 + \gamma x^3}{1 - x^3}.$$