

## Calculus — Homework 10 (Spring 2026)

1. Calculate the first order and second order partial derivatives.

(a)  $f(x, y) = 3x^2 - xy + y$ .

(c)  $f(x, y, z) = z \sin(x - y)$ .

(b)  $f(x, y) = x^2 e^{-y}$ .

(d)  $f(x, y, z) = z^{xy^2}$ .

2. Calculate.

(a) Find  $f_x(0, e)$ ,  $f_y(0, e)$ ,  $f_{xy}(0, e)$ ,  $f_{xxx}(0, e)$  and  $f_{xyx}(0, e)$  given that  $f(x, y) = e^x \ln y$ .

(b) Find  $f_x(0, \frac{1}{4}\pi)$ ,  $f_y(0, \frac{1}{4}\pi)$ ,  $f_{yy}(0, \frac{1}{4}\pi)$  and  $f_{xyxy}(0, \frac{1}{4}\pi)$  given that  $f(x, y) = e^{-x} \sin(x + 2y)$ .

(c) Find  $f_x(1, 2)$ ,  $f_y(1, 2)$  and  $f_{xx}(1, 2)$  given that  $f(x, y) = \frac{x}{x + y^2}$ .

3. Show that the functions  $u$  and  $v$  satisfy the **Cauchy–Riemann equations**

$$u_x(x, y) = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -v_x(x, y).$$

These equations are fundamentally important in the study of functions of a complex variable.

(a)  $u(x, y) = x^2 - y^2$ ;  $v(x, y) = 2xy$ . (Also compute  $(x + y\sqrt{-1})^2$ .)

(b)  $u(x, y) = e^x \cos y$ ;  $v(x, y) = e^x \sin y$ .

(c)  $u(x, y) = \frac{1}{2} \ln(x^2 + y^2)$ ;  $v(x, y) = \arctan \frac{y}{x}$ .

(d)  $u(x, y) = \frac{x}{x^2 + y^2}$ ;  $v(x, y) = \frac{-y}{x^2 + y^2}$ . (Also compute  $\frac{1}{(x + y\sqrt{-1})}$ .)

4. Let  $f(x, y) = x + xy$ . Use the definition of limits to prove that  $\lim_{(x,y) \rightarrow (1,0)} f(x, y) = 1$ .

5. Let  $g$  be a twice differentiable function of one variable and set

$$f(x, y) = g(x + y) + g(x - y).$$

Show that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}.$$

6. Let  $f$  be a smooth function of two variables. Show that

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x^2}.$$

7. Set

$$f(x, y) = \begin{cases} \frac{xy(y^2 - x^2)}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

(a) Find the second order partial derivatives of  $f$ .

(b) Show that

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0).$$

8. Find the gradient.

(a)  $f(x, y) = 3x^2 - xy + y$ .

(c)  $f(x, y, z) = z \sin(x - y)$ .

(b)  $f(x, y) = x^2 e^{-y}$ .

(d)  $f(x, y, z) = z^{xy^2}$ .

9. Find the gradient at  $\vec{p}$ .

(a)  $f(x, y) = 2x^2 - 3xy + 4y^2$ ;  $\vec{p} = (2, 3)$ .

(b)  $f(x, y) = 2x(x - y)^{-1}$ ;  $\vec{p} = (3, 1)$ .

(c)  $f(x, y, z) = e^{-x} \sin(z + 2y)$ ;  $\vec{p} = (0, \frac{1}{4}\pi, \frac{1}{4}\pi)$ .

(d)  $f(x, y, z) = \cos(xyz^2)$ ;  $\vec{p} = (\pi, \frac{1}{4}, -1)$ .