Calculus, Spring 2026, week 9

From now on, we will consider

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

In particular, we will consider

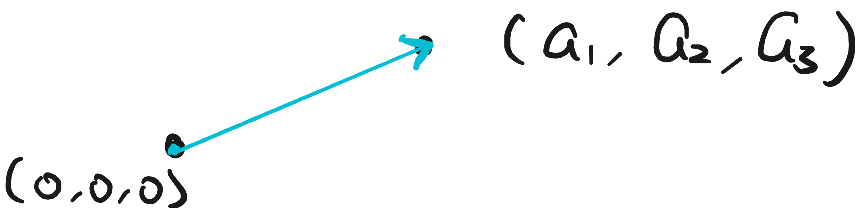
- $f: \mathbb{R} \to \mathbb{R}^3$ (or $\mathbb{R}^2$) —
  "curves in $\mathbb{R}^3$" or "vector-valued function"
- $f: \mathbb{R}^3 \to \mathbb{R}$ —
  "multivariable real-valued functions"

§ Vectors in $\mathbb{R}^3$

A point in $\mathbb{R}^3$, usually represented by

(a_1, a_2, a_3) or $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

where $a_1, a_2, a_3 \in \mathbb{R}$, can be regarded as



These vectors are equipped with the operations: "addition" and "scalar product"

Def (§13.2)

A vector $\vec{a}$ in $\mathbb{R}^3$ is an ordered triple of real numbers:

$$\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3.$$

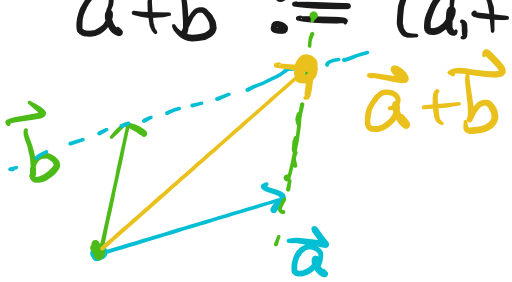
Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$.

(i) Equal:

$$\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$$

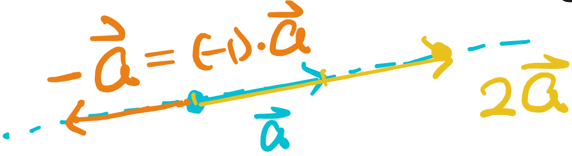
(ii) Addition:

$$\vec{a} + \vec{b} := (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$$



(iii) Scalar product: For $r \in \mathbb{R}$,

$$r \cdot \vec{a} := (ra_1,\ ra_2,\ ra_3)$$



Notations:

$$-\vec{a} = (-1)\cdot\vec{a}$$

$$\vec{a} - \vec{b} = \vec{a} + (-1)\cdot\vec{b}$$

$$\vec{0} = (0, 0, 0)$$

Prop

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$.

(i) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$

(ii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

(iii) $\vec{a}+\vec{0} = \vec{a}$

(iv) $\vec{a}+(-\vec{a}) = \vec{0}$

(v) $1\cdot\vec{a} = \vec{a}$

(vi) $\alpha\cdot(\vec{a}+\vec{b}) = \alpha\cdot\vec{a}+\alpha\cdot\vec{b}$

(vii) $(\alpha+\beta)\cdot\vec{\alpha} = \alpha\cdot\vec{a}+\beta\cdot\vec{b}$

Example

Let $\vec{a} = (1,-1,2)$, $\vec{b} = (2,3,-1)$, $\vec{c} = (0,1,0)$

① $\vec{a}-\vec{b} = (1-2, -1-3, 2-(-1))$

$= (-1,-4,3)$

② $2\vec{a}+3\vec{b}-\vec{c} = (2\cdot1+3\cdot2-0,$

$2\cdot(-1)+3\cdot3-1,$

$2\cdot2+3(-1)-0)$

$= (8,6,1)$

$2\vec{a}+2\vec{b}+\vec{b}-\vec{c} = 2(\vec{a}+\vec{b})+(\vec{b}-\vec{c})$

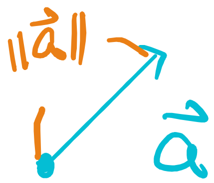
$= 2\cdot(3,2,1)+(2,2,-1)$

$= (8,6,1)$

Def

The norm (or magnitude, length) of $\vec{a}=(a_1,a_2,a_3)\in\mathbb{R}^3$ is

$$\|\vec{a}\| = \sqrt{a_1^2+a_2^2+a_3^2}$$



Prop

For $\vec{a},\vec{b}\in\mathbb{R}^3$, $r\in\mathbb{R}$,

(i) $\|\vec{a}\|\geq 0$, and $\|\vec{a}\|=0 \iff \vec{a}=\vec{0}$

(ii) $\|r\cdot\vec{a}\| = |r|\cdot\|\vec{a}\|$

(iii) $\|\vec{a}+\vec{b}\| \leq \|\vec{a}\|+\|\vec{b}\|$

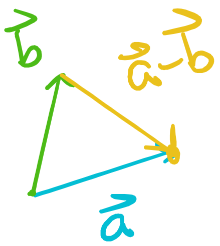
triangle inequality

Example

Let $\vec{a}=(1,-1,2)$, $\vec{b}=(2,3,-1)$

Then $\|\vec{a}\| = \sqrt{1^2+(-1)^2+2^2} = \sqrt{6}$

$\|\vec{b}\| = \sqrt{2^2+3^2+(-1)^2} = \sqrt{14}$



$\|\vec{a}+\vec{b}\| = \sqrt{(3,2,1)} = \sqrt{3^2+2^2+1^2}$

$= \sqrt{14} \leq \sqrt{6}+\sqrt{14} = \|\vec{a}\|+\|\vec{b}\|$

$\|\vec{a}-\vec{b}\| = \sqrt{26} \leq \|\vec{a}\|+\|-\vec{b}\| = \|\vec{a}\|+\|\vec{b}\|$

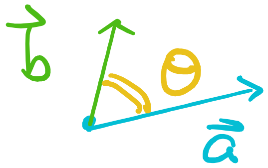
Def (Def 13.3.1)

For $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, we define the dot product by

$$\vec{a} \bullet \vec{b} := a_1b_1 + a_2b_2 + a_3b_2$$

Thm (Geometric meaning of $\vec{a}\bullet\vec{b}$)

Let θ be the angle

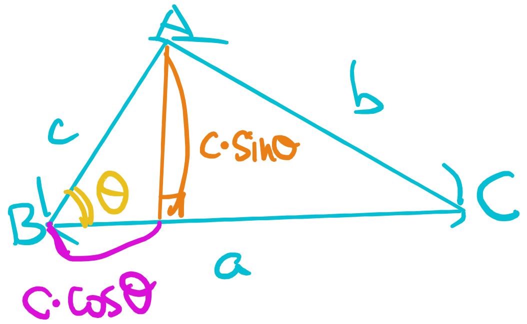


Then

$$\vec{a} \bullet \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$$

pf

Recall the law of cosine:



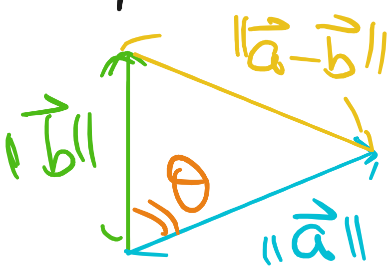
$$\Rightarrow b^2 = (c\cdot\sin\theta)^2 + (a - c\cdot\cos\theta)^2$$

$c^2 =$

$$= \boxed{c^2\cdot\sin^2\theta} + a^2 + \boxed{c^2\cos^2\theta} - 2ac\cos\theta$$

$$= a^2 + c^2 - 2ac\cos\theta$$

Apply it to



$(a = \|\vec{a}\|,\ c = \|\vec{b}\|)$

$$\Rightarrow \|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\theta$$

$$\Rightarrow \ \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\theta = \frac{1}{2}\left(\|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2\right)$$

$$= \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - (a_1-b_1)^2 - (a_2-b_2)^2 - (a_3-b_3)^2\right)$$

$$= a_1b_1 + a_2b_2 + a_3b_2 = \vec{a}\bullet\vec{b} \qquad \#$$

Notation:

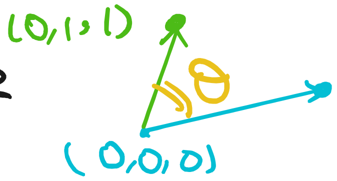
$\vec{a} \perp \vec{b}$ means $\vec{a}$ and $\vec{b}$ are perpendicular 垂直

$$\Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \vec{a}\bullet\vec{b} = 0$$

Example

① $(2,1,1) \perp (1,1,-3)$ since

$$2\cdot 1 + 1\cdot 1 + 1\cdot(-3) = 0$$

② The angle 

$(1,1,0)$ is

$$\theta = \cos^{-1}\left(\frac{0\cdot 1 + 1\cdot 1 + 1\cdot 0}{\sqrt{0^2+1^2+1^2}\cdot\sqrt{1^2+1^2+0^2}}\right) = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \qquad \#$$

Prop (§13.3) $\quad \vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$

(i) $\vec{a}\bullet\vec{a} = \|\vec{a}\|^2$

(ii) $\vec{a}\bullet\vec{b} = \vec{b}\bullet\vec{a}$

(iii) $\vec{a}\bullet(\alpha\vec{b} + \beta\vec{c}) = \alpha(\vec{a}\bullet\vec{b}) + \beta(\vec{a}\bullet\vec{c})$

e.g. $3\vec{a}\cdot(\vec{b}+4\vec{c}) = 3\vec{a}\cdot\vec{b}+(3\vec{a})\cdot(4\vec{c})$
$= 3\,\vec{a}\cdot\vec{b}+12\,\vec{a}\cdot\vec{c}$

Remark

① All the definitions and properties about vectors can be established for $\mathbb{R}^n$.

② In the textbook,

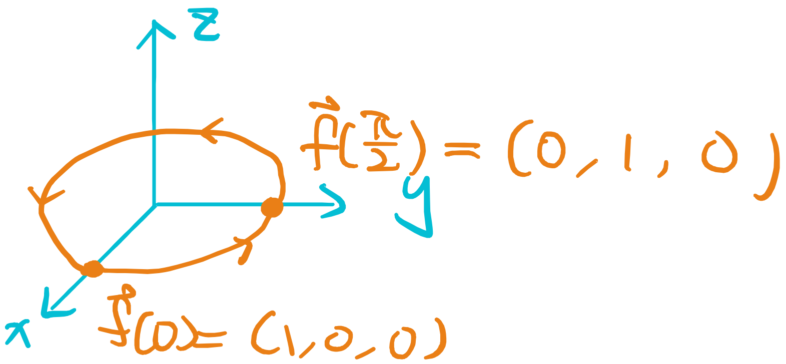
$$i = \vec{i} = (1,0,0)$$
$$j = \vec{j} = (0,1,0)$$
$$k = \vec{k} = (0,0,1)$$
$$\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

§ Limit and vector derivatives

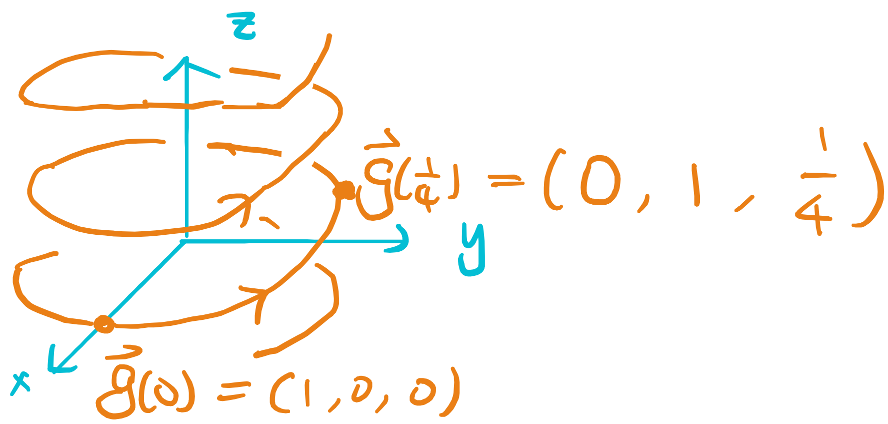
Consider
$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^3$$
Such a function can be expressed
$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$$
$$= f_1(t)\,\vec{i} + f_2(t)\,\vec{j} + f_3(t)\,\vec{k}$$
and can be considered a curve in $\mathbb{R}^3$

Example

① $\vec{f}(t) = \cos t \cdot \vec{i} + \sin t \, \vec{j} = (\cos t, \sin t, 0)$



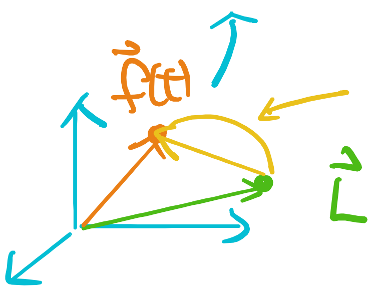
② $\vec{g}(t) = (\cos(2\pi t), \sin(2\pi t), t)$



Def (Def 14.1.1)

Let $\vec{f}: \mathbb{R} \to \mathbb{R}^3$, $t_0 \in \mathbb{R}$. We say the limit $\lim_{t \to t_0} \vec{f}(t)$ exists if $\exists \, \vec{L} \in \mathbb{R}^3$ s.t.

$$\lim_{t \to t_0} \|\vec{f}(t) - \vec{L}\| = 0$$



$\|\vec{f}(t) - \vec{L}\|$ = distance between $\vec{f}(t)$ and $\vec{L}$

In this case, we write $\lim_{t\to t_0} \vec{f}(t) = \vec{L}$.

Thm (14.1.4)

Let $\vec{L} = (L_1, L_2, L_3) \in \mathbb{R}^3$,

$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$

Then

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L}$$

$$\Longleftrightarrow \quad \lim_{t\to t_0} f_1(t) = L_1, \quad \lim_{t\to t_0} f_2(t) = L_2,$$

$$\lim_{t\to t_0} f_3(t) = L_3$$

pf

Note that, by definition,

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L}$$

$\| \vec{f}(t) - \vec{L} \|$

$\|$

$$\Longleftrightarrow \quad \lim_{t\to t_0} \sqrt{(f_1(t)-L_1)^2 + (f_2(t)-L_2)^2 + (f_3(t)-L_3)^2} = 0$$

pf of "$\Rightarrow$":

$i = 1, 2, 3$

Since

$$0 \leq | f_i(t) - L_i | = \sqrt{(f_i(t) - L_i)^2}$$

$$\leq \sqrt{(f_1(t)-L_1)^2+(f_2(t)-L_2)^2+(f_3(t)-L_3)^2}$$

$\to 0$ as $t \to t_0$

by the pinching thm,

$$\lim_{t\to t_0} |f_i(t)-L_i| = 0$$

$$\Longleftrightarrow \lim_{t\to t_0} f_i(t) = L_i \quad , \quad i=1,2,3.$$

pf of "$\Leftarrow$"

If

$$\lim_{t\to t_0} f_i(t) = L_i \quad , \quad i=1,2,3$$

then

$$\lim_{t\to t_0}\sqrt{(f_1(t)-L_1)^2+(f_2(t)-L_2)^2+(f_3(t)-L_3)^2}$$

$$= \sqrt{\left(\lim_{t\to t_0}(f_1(t)-L_1)\right)^2+\left(\lim_{t\to t_0} f_2(t)-L_2\right)^2+\left(\lim_{t\to t_0} f_3(t)-L_3\right)^2}$$

$$= \sqrt{0^2+0^2+0^2} = 0$$

\#

Example

Let $\vec{f}(t) = \left(\cos(t+\pi) \;,\; \sin(t+\pi) \;,\; e^{-t^2}\right)$

Then $\lim_{t\to 0} \vec{f}(t) = \left(\lim_{t\to 0}\cos(t+\pi) \;,\; \lim_{t\to 0}\sin(t+\pi) , \lim_{t\to 0} e^{-t^2}\right)$

$$= (\cos(0+\pi),\ \sin(0+\pi),\ e^{-0^2})$$

$$= (-1, 0, 1)$$

#

<u>Thm</u> (Thm 14.1.3)

Let $\vec{f}, \vec{g} : \mathbb{R} \to \mathbb{R}^3$, $u : \mathbb{R} \to \mathbb{R}$. $\alpha, \beta \in \mathbb{R}$

Suppose

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L},\quad \lim_{t\to t_0} \vec{g}(t) = \vec{M},\quad \lim_{t\to t_0} u(t) = A$$

Then

$$\lim_{t\to t_0} \left(\alpha \cdot \vec{f}(t) + \beta \cdot \vec{g}(t)\right) = \alpha \vec{L} + \beta \vec{M}$$

$$\lim_{t\to t_0} \left(u(t) \cdot \vec{f}(t)\right) = A \vec{L},$$

$$\lim_{t\to t_0} \|\vec{f}(t)\| = \|\vec{L}\|$$

$$\lim_{t\to t_0} \vec{f}(t) \bullet \vec{g}(t) = \vec{L} \bullet \vec{M}$$

$$\text{LHS} = \lim_{t\to t_0} \left(f_1(t) g_1(t) + f_2(t) g_2(t) + f_3(t) g_3(t)\right)$$

$$= \underbrace{\lim_{t\to t_0} f_1(t)}_{= L_1} \cdot \underbrace{\lim_{t\to t_0} g_1(t)}_{= M_1} + \underbrace{\lim_{t\to t_0} f_2(t)}_{= L_2} \cdot \underbrace{\lim_{t\to t_0} g_2(t)}_{= M_2} + \underbrace{\lim_{t\to t_0} f_3(t)}_{= L_3} \cdot \underbrace{\lim_{t\to t_0} g_3(t)}_{= M_3}$$

$\vec{L} = (L_1, L_2, L_3)$

$\vec{M} = (M_1, M_2, M_3)$

$= L_1M_1 + L_2M_2 + L_3M_3 = \vec{L} \cdot \vec{M} = RHS$

Def

A function $\vec{f}: \mathbb{R} \to \mathbb{R}^3$ is continuous at $c \in \mathbb{R}$ if

$$\lim_{t \to c} \vec{f}(t) = \vec{f}(c).$$

Remark

If $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$, then

$$\lim_{t \to c} \vec{f}(t) = \left(\lim_{t \to c} f_1(t), \lim_{t \to c} f_2(t), \lim_{t \to c} f_3(t)\right)$$

$$= \vec{f}(c) = (f_1(c), f_2(c), f_3(c))$$

$$\Leftrightarrow \begin{cases} \lim_{t \to c} f_1(t) = f_1(c) \\ \lim_{t \to c} f_2(t) = f_2(c) \\ \lim_{t \to c} f_3(t) = f_3(c) \end{cases}$$

$\Leftrightarrow$ f_1, f_2, f_3 are continuous at $t = c$

Def (Def 14.1.5)

A vector-value function $\vec{f}$ is differentiable at t if

$$\lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} \quad \text{exists}$$

If the limit exists, it is called the derivative of $\vec{f}$ at t, denoted by

$$\vec{f}'(t) \quad \text{or} \quad \frac{d\vec{f}}{dt}.$$

Thm (page 697)

Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ be a vector-valued function. Then $\vec{f}$ is differentiable at t iff f_1, f_2, f_3 are all differentiable at t

In this case,

$$\vec{f}'(t) = (f_1'(t), f_2'(t), f_3'(t))$$

Pf

By definition,

$$\vec{f}'(t) = \lim_{h\to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} \quad = \frac{1}{h}\cdot(\vec{f}(t+h) - \vec{f}(t))$$

$$= \lim_{h\to 0} \left(\frac{f_1(t+h) - f_1(t)}{h}, \frac{f_2(t+h) - f_2(t)}{h}, \frac{f_3(t+h) - f_3(t)}{h} \right)$$

$$= (f_1'(t), f_2'(t), f_3'(t))$$

#

Example

① If $\vec{f}(t) = (1,2,3)$ is constant, then

$$\vec{f}'(t) = \left((1)', (2)', (3)'\right)$$
$$= (0, 0, 0) = \vec{0} \quad \forall t \in \mathbb{R}.$$

② If $\vec{g}(t) = (t, t^2, -e^t)$, then

$$\vec{g}'(t) = \left((t)', (t^2)', (-e^t)'\right)$$
$$= (1, 2t, -e^t) \quad \forall t \in \mathbb{R}$$

\#

Cor $\Rightarrow$ f_1, f_2, f_3 differentiable at c $\Rightarrow$ they're continuous at c

If $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ is differentiable at $t=c$, then $\vec{f}(t)$ is continuous at $t=c$.

Thm (§14.2)

Let $\vec{f}(t)$ $\vec{g}(t)$ be differentiable vector-valued functions, $u(t)$ be a differentiable real-valued function, $\alpha, \beta \in \mathbb{R}$.

Then

(i) $\left(\alpha \vec{f}(t) + \beta \vec{g}(t)\right)' = \alpha \vec{f}'(t) + \beta \vec{g}'(t)$

(ii) $(u(t)\vec{f}(t))' = u'(t)\vec{f}(t) + u(t)\vec{f}'(t)$

(iii) $(\vec{f}(t)\cdot\vec{g}(t))' = \vec{f}'(t)\cdot\vec{g}(t)$
$\qquad + \vec{f}(t)\cdot\vec{g}'(t)$

(iv) $(\vec{f}(u(t)))' = (\vec{f}\circ u)'(t)$
$= \text{“}\vec{f}'(u(t))\cdot u'(t)\text{”}$
$= u'(t)\,\vec{f}'(u(t))$

chain rule

pf

(ii) $(u(t)\vec{f}(t))' = (u(t)f_1(t),\ u(t)f_2(t),\ u(t)f_3(t))'$

$u'(t)f_3(t) + u(t)f_3'(t)$

$= ((u(t)f_1(t))',\ (u(t)f_2(t))',\ (u(t)f_3(t))')$

$u'(t)f_1(t) + u(t)f_1'(t) \qquad u'(t)f_2(t) + u(t)f_2'(t)$

$= u'(t)(f_1(t), f_2(t), f_3(t))$
$\qquad + u(t)(f_1'(t), f_2'(t), f_3'(t))$

$= u'(t)\vec{f}(t) + u(t)f'(t)$ #

(iv) $(\vec{f}(u(t)))' = (f_1(u(t))',\ f_2(u(t))',\ f_3(u(t))')$

chain rule

$f_1'(u(t))\,u'(t) \qquad f_2'(u(t))\cdot u'(t) \qquad f_3'(u(t))\cdot u'(t)$

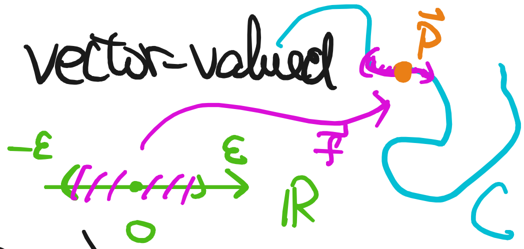
$= u'(t)\cdot(f_1'(u(t)), f_2'(u(t)), f_3'(u(t))) = u'(t)\vec{f}(t)$

#

§ Geometry of curves

A (differentiable) curve in $\mathbb{R}^3$ is a subset $C \subseteq \mathbb{R}^3$ satisfying the following condition:

$\forall \vec{p} \in C$, $\exists$ differentiable vector-valued function



$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t)), \quad t \in (-\varepsilon, \varepsilon)$$

s.t.

(i) $\vec{f}(t) \in C \quad \forall t \in (-\varepsilon, \varepsilon)$;

(ii) $\vec{f}(0) = \vec{p}$;

(iii) $\vec{f}'(0) \neq \vec{0}$ $\quad$ ($\Rightarrow$ $\vec{f}$ is NOT constant) $\quad \mathrm{im}(\vec{f}) \neq \{\bullet\}$

Such a $\vec{f}(t)$ is called a local parametrization of C near $\vec{p}$
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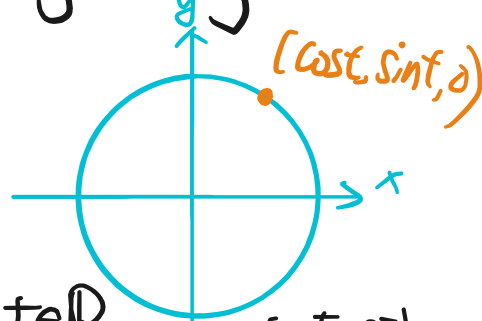
For example, the unit circle

$$S^1 \coloneqq \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$$

can be described by the parametrizations



$$\vec{f}(t) = (\cos t, \sin t, 0), \quad t \in \mathbb{R} \text{ or } t \in [0, 2\pi]$$

$$\vec{g}(t) = (\cos 2\pi t, \sin 2\pi t, 0), \quad t \in \mathbb{R}$$

In this context, a differentiable vector-valued is sometimes called a differentiable parametrized curve

Def (Def 14.3.1)

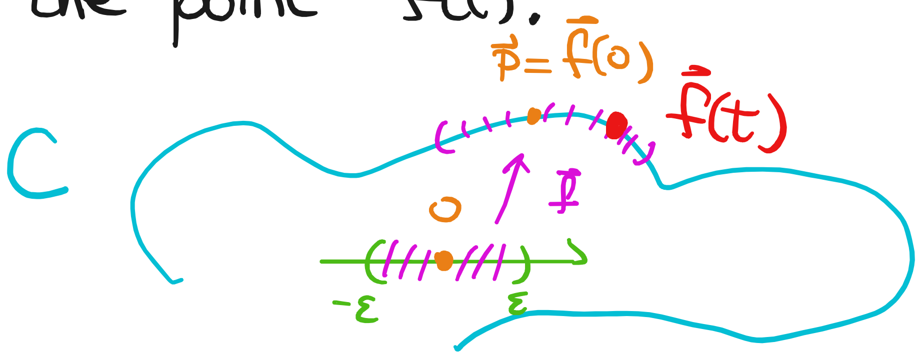
Let C be a curve in $\mathbb{R}^3$. A vector $\vec{v} \in \mathbb{R}^3$ is called a 切向量 tangent vector of C at $\vec{p} \in C$ if $\exists$ differentiable function $\vec{f} : (-\varepsilon, \varepsilon) \to C$ s.t.

$$\vec{f}(0) = \vec{p} \quad \text{and} \quad \vec{f}'(0) = \vec{v}$$

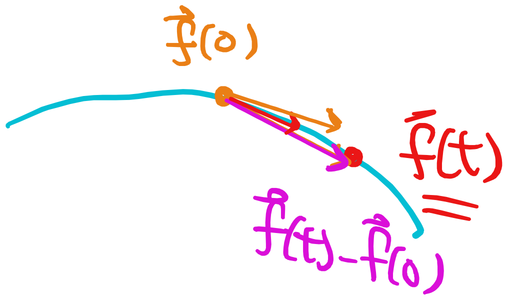
Similarly, for a differentiable parametrized curve $\vec{f}(t) = (f_1(t), f_2(t), f_2(t))$, the vector

$$\vec{f}'(t) = (f_1'(t), f_2'(t), f_3'(t))$$

is called a tangent vector of $\mathrm{im}(\vec{f})$ at the point $\vec{f}(t)$.



$$\vec{f}'(0) = \lim_{t\to 0} \frac{\vec{f}(0+t) - \vec{f}(0)}{t}$$

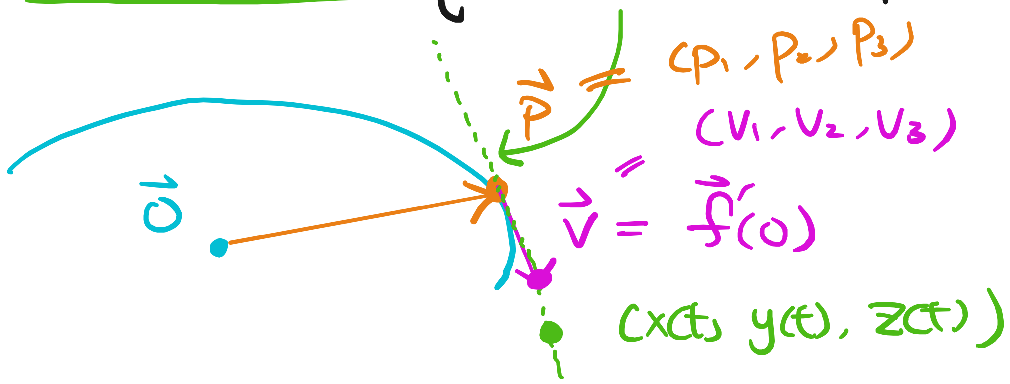


$$T_{\vec{p}}C := \{\text{tangent vectors of } C \text{ at } \vec{p} \in C\}$$
$$= \left\{ \vec{f}'(0) \;\middle|\; \begin{array}{l} \vec{f} : (-\varepsilon, \varepsilon) \to C \text{ differentiable} \\ \vec{f}(0) = \vec{p} \end{array} \right\}$$

切线

The tangent line of C at $\vec{p}$ is

$$\vec{p} + T_{\vec{p}}C = \{\vec{p} + \vec{v} \mid \vec{v} \in T_{\vec{p}}C\}$$



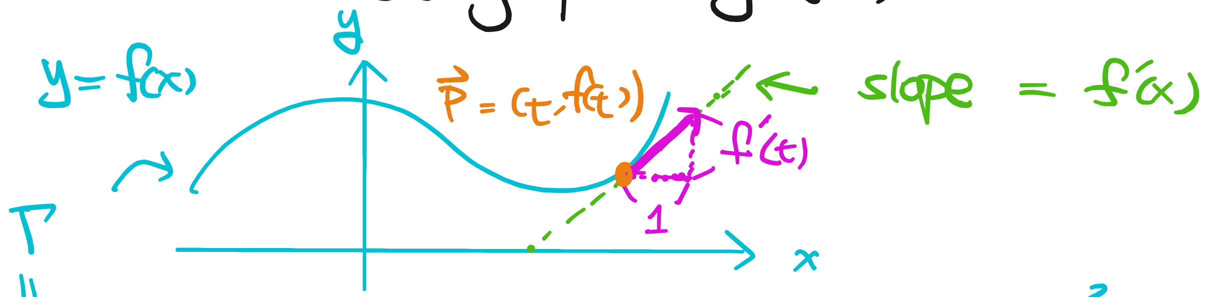
Remark: A parametriation of the tangent line is

$$\begin{cases} x(t) = p_1 + v_1 \cdot t \\ y(t) = p_2 + v_2\, t \\ z(t) = p_3 + v_3\, t \end{cases}$$

Remark

Recall if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then

$f'(x)$ = slope of the tangent line of the graph $y = f(x)$



$\Gamma = \{(x, f(x)) \mid x \in \mathbb{R}\}$ is a curve in $\mathbb{R}^2 \subseteq \mathbb{R}^3$

Note that Γ can be parametrized by

$$\vec{\gamma}(t) = (t, f(t))$$

$\Rightarrow$ $\vec{\gamma}'(t) = (1, f'(t))$ is a tangent vector of Γ at $(t, f(t)) = \vec{p}$

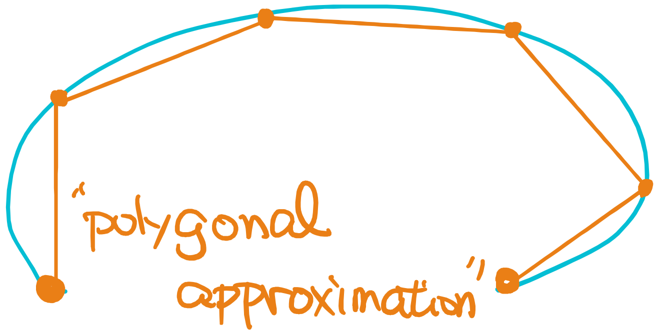
$\Rightarrow$ the slope of the tangent line is

$$\frac{f'(t)}{1} = f'(t)$$

which matches with what we learned in the last semester.

Application to arc length

The arc length (弧長 = 曲線長) of a curve is the limit of the lengths of inscribed polygonal approximations



We have the following formula

<u>Thm</u> (Thm 14.4.2)

Let $\vec{f}(t)$, $t \in [a,b]$, be a continuously differentiable parametrized curve in $\mathbb{R}^3$. The arc length from $\vec{f}(a)$ to $\vec{f}(b)$ is given by L

$$L = \int_a^b \|\vec{f}'(t)\| \, dt$$

<u>sketch of pf</u>

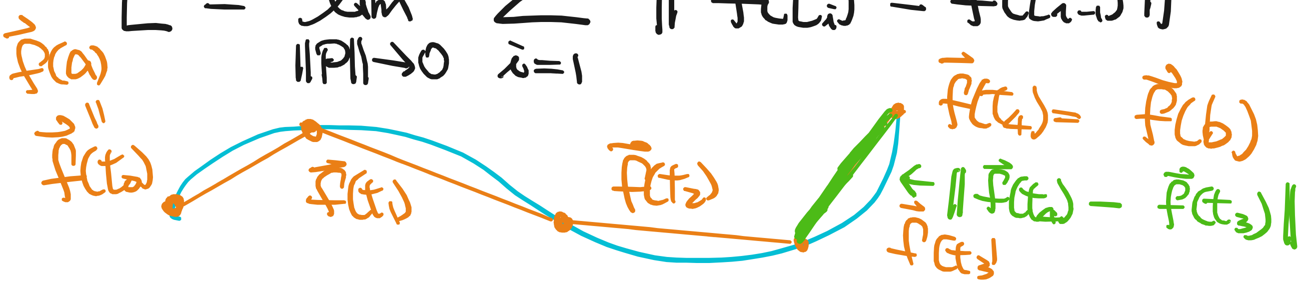
Let

$$P = \{ a = t_0 < t_1 < \cdots < t_n = b \}$$

be a partition of $[a,b]$.

By definition, the arc length is

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \| \vec{f}(t_i) - \vec{f}(t_{i-1}) \|$$



By Mean Value Thm, $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$

$$f_1(t_i) - f_1(t_{i-1}) = f_1'(\tau_{i,1}) \cdot (t_i - t_{i-1})$$
$$f_2(t_i) - f_2(t_{i-1}) = f_2'(\tau_{i,2}) \cdot (t_i - t_{i-1})$$
$$f_3(t_i) - f_3(t_{i-1}) = f_3'(\tau_{i,3}) \cdot (t_i - t_{i-1})$$

for some $\tau_{i,j} \in (t_{i-1}, t_i)$

$\Rightarrow$ $L =$

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} \underbrace{\sqrt{(f_1'(\tau_{i,1}))^2 + (f_2'(\tau_{i,2}))^2 + (f_3'(\tau_{i,3}))^2} \cdot |t_i - t_{i-1}|}_{\approx \|\vec{f}'(\tau_i)\| \cdot (t_i - t_{i-1})}$$

skip

$$= \lim_{\|P\|\to 0} \sum_{i=1}^{n} \|\vec{f}'(\tau_i)\| \cdot (t_i - t_{i-1})$$

= Riemann sum of $\|\vec{f}(t)\|$

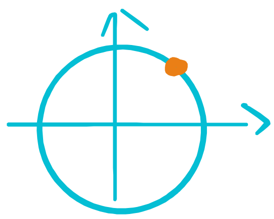
where $\tau_i \in (t_{i-1}, t_i)$

$$= \int_a^b \|\vec{f}'(t)\| \, dt$$

#

Example

①



$\Rightarrow \vec{f}'(t) = (-\sin t, \cos t, 0)$

$\vec{f}(t) = (\cos t, \sin t, 0)$, $t \in [0, 2\pi]$

$\vec{g}'(t) = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t, 0)$

$\vec{g}(t) = (\cos 2\pi t, \sin 2\pi t, 0)$, $t \in [0, 1]$

By Thm, the arc length of unit circle

$$= \int_0^{2\pi} \|\vec{f}(t)\| dt = \int_0^{2\pi} \overbrace{\sqrt{(-\sin t)^2 + (\cos t)^2 + 0^2}}^{=1} \, dt = 2\pi$$

$$= \int_0^1 \| \vec{g}'(t) \| \, dt = \int_0^1 \underbrace{\sqrt{4\pi^2(\sin 2\pi t)^2 + 4\pi^2(\cos 2\pi t)^2}}_{= 2\pi} \, dt$$

$$= 2\pi \qquad \#$$

② Find the length of the curve

$$\vec{f}(t) = (\cos t, \ \sin t, \ t) \qquad t \in [0, 2\pi]$$

$\Rightarrow \vec{f}'(t) = (-\sin t, \cos t, 1)$

length $= \displaystyle\int_0^{2\pi} \| \vec{f}'(t) \| \, dt$

$$= \int_0^{2\pi} \underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}}_{= \sqrt{2}} \, dt$$

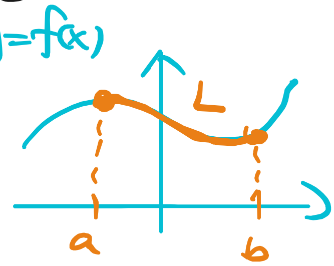
$$= 2\sqrt{2}\,\pi \qquad \#$$

Remark

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function

Then the arc length L of $y = f(x)$ from a to b is:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$



because

$$\vec{\gamma}(t) = (t, f(t)), \qquad t \in [a, b]$$

$\Rightarrow \vec{\gamma}'(t) = (1, f'(t))$

is a parametrization of the curve $y=f(x)$.

So, by Thm,

$$L = \int_a^b \| \vec{\gamma}'(t) \| \, dt$$

$$= \int_a^b \sqrt{1^2 + (f'(t))^2} \, dt$$