Calculus, Spring 2026, week 8

§ Bernoulli polynomial

Motivating question:

$$1^k + 2^k + \cdots + n^k = ?$$

e.g. $1+2+\cdots+n = \frac{n(n+1)}{2}$

Suppose $\exists$ polynomial $B_k(x)$ of deg k s.t.

$$\int_x^{x+1} B_k(t)\,dt = \frac{x^k}{k!} \qquad \forall x \in \mathbb{R}$$

Then

$$\underbrace{0^k}_{=\,k!\int_0^1 B_k(t)dt} + \overbrace{1^k}^{=\,k!\int_1^2 B_k(t)dt} + \underbrace{2^k}_{=\,k!\int_2^3 B_k(t)dt} + \cdots + n^k = k! \cdot \int_0^{n+1} B_k(t)\,dt$$

<u>Def</u>

The k-th Bernoulli polynomial $B_k(x)$ is the polynomial of deg k s.t.

$$(*) \quad \int_x^{x+1} B_k(t)\,dt = \frac{x^k}{k!} \qquad \forall x \in \mathbb{R}.$$

Basic questions:

$\exists !$ such $B_k(x)$?

<u>Lemma</u>

There is at most one polynomial of deg k satisfying $(*)$

pf

Suppose $P(x)$ and $Q(x)$ are polynomials of deg k satisfying ⊛.

Let $D(x) = P(x) - Q(x)$.

$$\Rightarrow \int_x^{x+1} D(t)\,dt = \int_x^{x+1} P(t)\,dt - \int_x^{x+1} Q(x)\,dt$$

$$\overset{⊛}{=} \frac{x^k}{k!} - \frac{x^k}{k!} = 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow 0 = \left(\int_x^{x+1} D(t)\,dt \right)' = D(x+1) - D(x) \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ $D(x) - D(0)$ is a polynomial that has ∞ zeros: $D(m) - D(0) = 0 \quad \forall m \in \mathbb{N}$.

$$\Rightarrow D(x) - D(0) = 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ $D(x) = D(0)$ is a constant polynomial

Since

$$\int_x^{x+1} D(t)\,dt = 0 = \int_x^{x+1} D(0)\,dt = D(0)$$

we have

$$P(x) - Q(x) = D(x) = 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \quad P(x) = Q(x) \qquad \#$$

Lemma

For every $k \in \mathbb{N} \cup \{0\}$, $\exists$ polynomial $B_k(x)$ of deg k satisfying ⊛

pf (Induction on k)

<u>$k=0$</u>: Choose $B_0(x) = 1$

$$\Rightarrow \int_x^{x+1} B_0(t)\,dt = \int_x^{x+1} 1\,dt = 1 = \frac{x^0}{0!}$$

<u>Inductive step</u>:

Assume $\exists$ polynomial $B_{k-1}(x)$ of deg $k-1$ s.t.

$$\int_x^{x+1} B_{k-1}(t)\,dt = \frac{x^{k-1}}{(k-1)!}$$

Observation:

Want $B_k(x)$ s.t.

$$\circledast \quad \int_x^{x+1} B_k(t)\,dt = \frac{x^k}{k!} \qquad \forall x \in \mathbb{R}$$

$$\Rightarrow \left(\int_x^{x+1} B_k(t)\,dt\right)' = \left(\frac{x^k}{k!}\right)'$$

$$\circledast\circledast \quad B_k(x+1) - B_k(x) = \frac{x^{k-1}}{(k-1)!} = \int_x^{x+1} B_{k-1}(t)\,dt$$

To satisfy $\circledast\circledast$, define

$$B_k(x) = \int_0^x B_{k-1}(t)\,dt + C_n$$

where $C_n \in \mathbb{R}$.

To satisfy $\circledast$, at $x=0$, we have

$$\int_0^1 B_k(t)\,dt = 0$$

$$\int_0^1 \left(\int_0^t B_{k-1}(s)\,ds + C_n\right) dt = \int_0^1 \int_0^t B_{k-1}(s)\,ds\,dt + C_n$$

$$\Rightarrow C_n = -\int_0^1 \int_0^t B_{k-1}(s)\,ds\,dt \in \mathbb{R}$$

Check $B_k(x)$ is what we want:

① Since $B_{k-1}(x)$ is a polynomial of deg $k-1$,

$$B_k(x) = \int_0^x B_{k-1}(t)\,dt + C_n \quad \text{is a poly of deg } k$$

② Let

$$C(x) = \int_x^{x+1} B_k(t)\,dt$$

$$\Rightarrow C'(x) = B_k(x+1) - B_k(x) = \int_x^{x+1} B_{k-1}(t)\,dt = \frac{x^{k-1}}{(k-1)!}$$

and

$$C(0) = \int_0^1 B_k(t)\,dt = 0$$

$$\Rightarrow C(x) = \int_0^x C'(t)\,dt = \int_0^x \frac{t^{k-1}}{(k-1)!}\,dt = \frac{t^k}{k!}$$

So $\displaystyle\int_x^{x+1} B_k(t)\,dt = \frac{x^k}{k!}$ #

<u>Conclusion</u>: $\exists!$ polynomial $B_k(x)$ of deg k satisfying

$$\int_x^{x+1} B_k(t)\,dt = \frac{x^k}{k!}$$

Generating function for Bernoulli polynomials

$$G(s,t) := \sum_{k=0}^{\infty} B_k(s) \cdot t^k \cancel{\frac{1}{k!}}$$

Remark

① $\int_x^{x+1} B_k'(t)\,dt = B_k(x+1) - B_k(x) = \frac{d}{dx}\left(\int_x^{x+1} B_k(t)\,dt\right) = \frac{d}{dx}\left(\frac{x^k}{k!}\right) = \frac{x^{k-1}}{(k-1)!}$

$\Rightarrow \boxed{B_k'(t) = B_{k-1}(t)}$

$\Rightarrow \frac{d}{ds}G(s,t) \text{ "=" } \sum_{k=1}^{\infty} B_k'(s) t^k = \sum_{k=1}^{\infty} B_{k-1}(s) t^k = t \cdot G(s,t)$

$\Rightarrow \boxed{G(s,t) = a(t) e^{st}}$

② $\int_x^{x+1} G(s,t)\,ds \text{ "=" } \sum_{k=0}^{\infty} \int_x^{x+1} B_k(s)\,ds\, t^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} t^k = e^{xt}$

$\frac{d}{dx} \Rightarrow G(x+1,t) - G(x,t) = t e^{xt}$

$a(t) e^{(x+1)t} - a(t) e^{xt} = a(t)\cdot(e^t - 1) e^{xt}$

$\Rightarrow \boxed{a(t) = \frac{t}{e^t - 1}}$

Thm For $|t| < 2\pi$,

$$G(s,t) = \frac{t e^{st}}{e^t - 1}$$

pf: skip

Convention 1:

$$\int_x^{x+1} B_k(t)\,dt = \frac{x^k}{k!}$$

$$G(s,t) = \sum_{k=0}^{\infty} B_k(s)\, t^k = \frac{t e^{st}}{e^t - 1}$$

Convention 2:

$$\int_x^{x+1} \mathcal{B}_k(t)\,dt = x^k$$

$$G(s,t) = \sum_{k=0}^{\infty} \mathcal{B}_k(s) \cdot \frac{t^k}{k!} = \frac{t e^{st}}{e^t - 1}$$

To analyze $G(s,t)$, we need:

Note: HW6 Problem 6. is corrected!!

<u>exercise</u>

Prove that if $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are absolutely convergent, then so is $\sum_{n=0}^{\infty} c_n$ and

$$\left(\sum_{k=0}^{\infty} a_k\right)\cdot\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty} c_n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$(a_0+a_1+a_2+\cdots)(b_0+b_1+b_2+\cdots)$$

$$\begin{aligned} = a_0b_0 &+ a_0b_1 + a_0b_2 + \cdots \\ &+ a_1b_0 + a_1b_1 + a_1b_2 + \cdots \\ &\qquad\quad + a_2b_0 + a_2b_1 + \cdots \end{aligned}$$

$$c_0 + c_1 + c_2 + \cdots$$

Let $B_k := B_k(0)$, called <u>Bernoulli numbers</u>

Since $G(s,t) = \sum_{k=0}^{\infty} B_k(s) \frac{t^k}{\cancel{k!}} \overset{\text{Thm}}{\underset{|t|<2\pi}{=}} \frac{te^{st}}{e^t-1}$

we have $(s=0)$

$$\sum_{k=0}^{\infty} B_k \frac{t^k}{\cancel{k!}} = \frac{t}{e^t-1} \qquad \sum_{j=0}^{\infty} \frac{t^j}{j!}$$

$$\Rightarrow t = \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{\cancel{k!}}\right)\cdot(e^t-1) = \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{\cancel{k!}}\right)\left(\sum_{j=1}^{\infty} \frac{t^j}{j!}\right)$$

$$\underline{\text{exer}} = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} \frac{B_k}{\cancel{k!}} \cdot \frac{1 \; n!}{(n-k)!}\right)\frac{t^n}{n!}$$

$$= C^n_k = \binom{n}{k}\cdot k!$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} k!\binom{n}{k} B_k\right)\cdot \frac{t^n}{n!}$$

Compare coefficients:

$n=1$:
$$1 = \sum_{k=0}^{0} k!\binom{1}{k} B_k = B_0$$

$n>1$:
$$0 = \sum_{k=0}^{n-1} k!\binom{n}{k} B_k \;\xleftarrow{\times \frac{1}{n!}}$$

遞迴關係

or
$$0 = \sum_{k=0}^{n-1} \frac{B_k}{(n-k)!}$$

$\Rightarrow$ We can compute B_k:

$B_0 = 1$,

$n=2$:
$$0 = \sum_{k=0}^{1} k!\binom{2}{k} B_k = 1\cdot B_0 + 1\cdot 2B_1 \Rightarrow B_1 = -\frac{1}{2}$$

$n=3$:
$$0 = \sum_{k=0}^{2} k!\binom{3}{k} B_k = 1\cdot \underset{1}{B_0} + 1\cdot 3\underset{\frac{1}{2}}{B_1} + 2\cdot 3B_2 \Rightarrow B_2 = \cancel{\tfrac{1}{6}}\ \frac{1}{12}$$

$n=4$:
$$0 = 1\cdot \underset{1}{B_0} + 1\cdot \underset{-2}{4B_1} + 2\cdot \underset{1}{6B_2} + 6\cdot 4B_3 \Rightarrow B_3 = 0$$

$$\vdots$$

Note that
$$G(s,t) = \sum_{n=0}^{\infty} B_n(s)\frac{t^n}{\cancel{n!}} = \frac{t}{e^t-1}e^{st}$$

$$= \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{\cancel{k!}}\right)\left(\sum_{j=0}^{\infty} \frac{(st)^j}{j!}\right)$$

$$\overset{\text{exer}}{=} \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{B_k}{\cancel{k!}} \cdot \frac{s^{n-k}}{(n-k)!}\right) t^n$$

$$\Rightarrow \quad \frac{B_n(s)}{\cancel{n!}} = \sum_{k=0}^{n} \frac{B_k}{\cancel{k!}} \frac{s^{n-k}}{(n-k)!}$$

$$\Rightarrow \quad B_n(s) = \sum_{k=0}^{n} \cancel{\binom{n}{k}} B_k \, s^{n-k} / (n-k)!$$

Using $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \cancel{\frac{1}{6}}\ \frac{1}{12}$, $B_3 = 0$:

$$B_0(s) = 1$$

$$B_1(s) = B_0 s^1 + B_1 = s - \frac{1}{2}$$

$$B_2(s) = \frac{1}{2!} B_0 s^2 + \cancel{2} B_1 \frac{s}{1!} + B_2$$

$$= \frac{1}{2} s^2 - \frac{s}{2} + \cancel{\frac{1}{6}}\ \frac{1}{12}$$

$$B_3(s) = \frac{1}{3!} B_0 s^3 + \cancel{3} B_1 \frac{s^2}{2!} + \cancel{3} B_2 \frac{s}{1!} + \frac{B_3}{0!}$$

$$= \frac{s^3}{6} - \cancel{\frac{3}{2}}\ \frac{s^2}{4} + \cancel{\frac{1}{2}}\ \frac{s}{12}$$

Back to the question

$$1^k + 2^k + \cdots + n^k = ?$$

Prop

$$1^k + 2^k + \cdots + n^k = \left(\frac{B_{k+1}(n+1) - B_{k+1}}{\cancel{k+1}}\right) \cdot k!$$

Pf

Since

$$G(s+1,t) - G(s,t) = \frac{te^{(s+1)t}}{e^t-1} - \frac{te^{st}}{e^t-1}$$

$$= te^{st}\,\frac{e^t-1}{e^t-1} = te^{st}$$

$$\Rightarrow \sum_{k=0}^{\infty}\left(B_k(s+1)-B_k(s)\right)\frac{t^k}{\cancel{k!}} = \sum_{k=1}^{\infty} \cancel{k}\frac{s^{k-1}}{(k-1)!}\,\frac{t^k}{\cancel{k!}} \overset{k=j+1}{=} \sum_{j=0}^{\infty}\frac{s^j t^{j+1}}{j!}$$

$$\Rightarrow B_{k+1}(s+1) - B_{k+1}(s) = \cancel{(k+1)}\frac{s^k}{k!}$$

$$\Rightarrow \sum_{s=0}^{n}\left(B_{k+1}(s+1) - B_{k+1}(s)\right) = \sum_{s=0}^{n}\cancel{(k+1)}\frac{s^k}{k!}$$

$$\Rightarrow 1^k + 2^k + \cdots + n^k = \left(\sum_{s=0}^{n}\frac{s^k}{k!}\right)\cdot k!$$

$$= \frac{1}{\cancel{k+1}}\,k!\cdot\Big(B_{k+1}(n+1) - \cancel{B_{k+1}(n)} + \cancel{B_{k+1}(n)} - \cancel{B_{k+1}(n-1)} + \cancel{\cdots} + \cancel{B_{k+1}(2)} - \cancel{B_{k+1}(1)} + \cancel{B_{k+1}(1)} - B_{k+1}(0)\Big)$$

$$= \left(\frac{B_{k+1}(n+1) - B_{k+1}}{\cancel{k+1}}\right)\cdot k! \qquad \#$$

Example $(k=2)$

By Prop,

$$1^2 + 2^2 + \cdots + n^2 = \left(\frac{B_3(n+1) - B_3}{\cancel{3}}\right)2!$$

Since

$$B_3 = 0, \qquad B_3(s) = \frac{s^3}{6} - \cancel{\tfrac{3}{2}}\frac{s^2}{4} + \cancel{\tfrac{1}{2}}\,s/12$$

We have

$$1^2+2^2+\cdots+n^2 = \cancel{\tfrac{1}{3}}\left(\frac{(n+1)^3}{6} - \cancel{\tfrac{3}{2}}\frac{(n+1)^2}{4} + \cancel{\tfrac{1}{2}}\frac{(n+1)}{12} - 0\right)3!$$

$$= \cdots = \frac{n(n+1)(2n+1)}{6} \qquad \#$$

§ Gamma function

Recall

$$n! = 1\cdot 2\cdots n \qquad \text{for } n \in \mathbb{N}\cup\{0\}$$

Can we define $(\frac{1}{2})!$?

Hope: find a continuous function

$$\Gamma : (0,\infty) \to \mathbb{R}$$

s.t. $\Gamma(n+1) = n!$

Def

For $x>0$, the Gamma function $\Gamma(x)$ is defined by the improper integral

$$\Gamma(x) := \int_0^\infty e^{-t}\, t^{x-1}\, dt.$$

Lemma

The improper integral $\int_0^\infty e^{-t} t^{x-1}\,dt$ converges.

pf

Recall:

① If $f(t) \in C[a,\infty)$, then $\int_a^\infty f(t)\,dt = \lim_{b\to\infty}\int_a^b f(t)\,dt.$

② If $f(t) \in C(a,b]$, then $\int_a^b f(t)dt = \lim_{c \to a^+} \int_c^b f(t)dt$

Note that

$$e^{-t}\, t^{x-1} \in C(0,\infty)$$

<0 if $x<1$

$$\Rightarrow \int_0^\infty e^{-t} t^{x-1} dt = \underbrace{\int_0^1 e^{-t} t^{x-1} dt}_{\text{type ②}} + \underbrace{\int_1^\infty e^{-t} t^{x-1} dt}_{\text{type ①}}$$

Convergence of ①:

$$\int_1^\infty e^{-t} t^{x-1} dt = \lim_{b \to \infty} \underbrace{\int_1^b e^{-t} t^{x-1} dt}_{=:F(b)}$$

Since $e^{-t} t^{x-1} > 0 \quad \forall\, t \geq 1$, the function $F: [1,\infty) \to \mathbb{R}$ is an increasing function.

Note that

use 羅必塔 if necessary

$$\lim_{t \to \infty} e^{-\frac{t}{2}} \cdot t^{x-1} = \lim_{t \to \infty} \frac{t^{x-1}}{e^{\frac{t}{2}}} = 0$$

$\Rightarrow$ For $\varepsilon = 1$, $\exists\, N = N_{\varepsilon, x}$ s.t.

$$\left| e^{-\frac{t}{2}} \cdot t^{x-1} - 0 \right| = e^{-\frac{t}{2}} t^{x-1} < \varepsilon = 1$$

$\forall\, t \geq N$

$$\Rightarrow \quad e^{-t} \cdot t^{x-1} = e^{-\frac{t}{2}} \cdot (e^{-\frac{t}{2}} t^{x-1}) < e^{-\frac{t}{2}}$$

$\forall\, t \geq N$

$\Rightarrow$ For $b \geq N$,

$\checkmark < e^{-\frac{t}{2}}$

$$\int_1^b e^{-t} t^{x-1} dt = \int_1^N e^{-t} t^{x-1} dt + \int_N^b e^{-t} t^{x-1} dt$$

$$\leq \int_1^N e^{-t} t^{x-1}\,dt + \underbrace{\int_N^b e^{-\frac{t}{2}}\,dt}_{}$$ $= (-2e^{-\frac{t}{2}})'$

$= -2e^{-\frac{t}{2}}\Big|_{t=N}^{b}$

$$= \int_1^N e^{-t} t^{x-1}\,dt - 2e^{-\frac{b}{2}} + 2e^{-\frac{N}{2}}$$

$$\leq \int_1^N e^{-t} t^{x-1}\,dt + 2e^{-\frac{N}{2}}$$ $\leftarrow$ indep. of b

$=$ an upper bound of $F(b)$.

$\Rightarrow \int_1^\infty e^{-t} t^{x-1}\,dt = \lim_{b\to\infty} F(b)$ Converges.

Convergence of ②: $(0<x<1)$

$$\int_0^1 e^{-t} t^{x-1}\,dt = \lim_{a\to 0^+} \int_a^1 \underbrace{e^{-t} t^{x-1}}_{>0}\,dt$$

$$= \lim_{c\to+\infty} \int_{\frac{1}{c}}^1 e^{-t} t^{x-1}\,dt = G(c)$$

Since $e^{-t} t^{x-1} > 0$, the function

$G: [1,\infty) \to \mathbb{R}$

is an increasing function. (Want to find an upper bound for $G(c)$)

For $0<t\leq 1$, since

$$e^{-t} t^{x-1} = \frac{t^{x-1}}{e^t} < t^{x-1}$$

$1 < e^t \leq e$

$(0<x<1)$

we have, for $c \in [1,\infty)$,

$$G(c) = \int_{\frac{1}{c}}^1 e^{-t} t^{x-1}\,dt \leq \int_{\frac{1}{c}}^1 t^{x-1}\,dt$$

$t^{x-1} = \left(\frac{t^x}{x}\right)'$

indep. of c

$$= \frac{t^x}{x}\Big|_{t=\frac{1}{c}}^{1} = \frac{1}{x} - \frac{1}{x}\cdot\frac{1}{c^x} < \frac{1}{x}$$

$\Rightarrow$ G is increasing and bounded above

$\Rightarrow$ $\int_0^1 e^{-t} t^{x-1} dt = \lim_{c\to\infty} G(c)$ exists (converges).

Conclusion:

$$\int_0^\infty e^{-t} t^{x-1} dt = \int_0^1 e^{-t} t^{x-1} dt + \int_1^\infty e^{-t} t^{x-1} dt$$

converges. $\Rightarrow$ $\Gamma(x)$ is well-defined #

<u>Thm</u>

For $x>0$,

$$\Gamma(x+1) = x\cdot\Gamma(x)$$

<u>pf</u>

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^{\cancel{(x+1)-1}\ x}\, dt$$

$\int u v' dt = uv - \int v u' dt$

$$= \lim_{b\to\infty} \int_0^b \underbrace{e^{-t}}_{=v'} \overset{=u}{t^x} dt$$

$u(t) = x\, t^{x-1}$

$v(t) = -e^{-t}$

$$= \lim_{b\to\infty}\Big(\underbrace{-e^{-t}\cdot t^x \Big|_{t=0}^{b}}_{= -\frac{b^x}{e^b} \to 0 \text{ as } b\to\infty} + x\int_0^b e^{-t}\cdot t^{x-1} dt \Big)$$

$$= \lim_{b\to\infty} x\cdot \int_0^b e^{-t} t^{x-1} dt = x \overbrace{\int_0^\infty e^{-t} t^{x-1} dt}^{=\Gamma(x)} = x\Gamma(x)$$

#

Cor

For $n \in \mathbb{N} \cup \{0\}$,

$$\Gamma(n+1) = n!$$

pf

$$\Gamma(1) \overset{(n=0)}{=} \int_0^\infty e^{-t} \cdot t^{1-1}\, dt = \int_0^\infty e^{-t}\, dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-t}\, dt = \lim_{b\to\infty} \big(\underbrace{-e^{-t}\big|_{t=0}^{b}}_{-\frac{1}{e^b} + 1} \big)$$

$$\frac{1}{e^b} \to 0$$

$$= 1 = 0!$$

$$\Gamma(n+1) \overset{\text{Thm}}{=} n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = n(n-1)(n-2)\,\Gamma(n-2)$$

$$= \cdots = \underbrace{\boxed{n(n-1)(n-2)\cdots \cdot 1}}_{= n!} \cdot \Gamma(1) = 1$$

$$= n! \qquad \#$$

Beta function

Def

For $x, y > 0$, the Beta function $B(x,y)$ is defined by the (improper) integral

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

Lemma

The integral $\int_0^1 t^{x-1}(1-t)^{y-1}\, dt$ converges for $x, y > 0$.

pf

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \underbrace{\int_0^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt}_{(\mathrm{I})} + \underbrace{\int_{\frac{1}{2}}^1 t^{x-1}(1-t)^{y-1}dt}_{(\mathrm{II})}$$

Convergence of (I):

$$\int_0^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt = \lim_{a\to 0^+}\int_a^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt$$

$$\overset{(a=\frac{1}{c})}{=} \lim_{c\to+\infty} \int_{\frac{1}{c}}^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt = F(c)$$

Since $t^{x-1}(1-t)^{y-1} > 0$ for $t\in(0,1)$,

the function $F: [2,+\infty) \to \mathbb{R}$ is increasing.

Since

$$\lim_{t\to 0^+}(1-t)^{y-1} = 1$$

for $\varepsilon = \frac{1}{2} > 0$, $\exists\ \frac{1}{2} > \delta > 0$ s.t.

$$\left|(1-t)^{y-1} - 1\right| < \frac{1}{2} \quad \text{for } 0<t<\delta$$

$$\Rightarrow \quad \frac{1}{2} < (1-t)^{y-1} < \frac{3}{2} \quad \text{for } 0<t<\delta$$

$$\Rightarrow \quad F(c) = \int_{\frac{1}{c}}^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt$$

$$= \int_{\frac{1}{c}}^{\delta} \underbrace{t^{x-1}(1-t)^{y-1}}_{< \frac{3}{2}t^{x-1}} dt + \int_{\delta}^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt$$

$$(\text{for } c > \tfrac{1}{\delta}) \leq \int_{\frac{1}{c}}^{\delta} \underbrace{\frac{3}{2} t^{x-1}}_{= \left(\frac{3}{2}\frac{t^x}{x}\right)'} dt + \int_{\delta}^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt$$

$$= \frac{3}{2}\frac{\delta^x}{x} - \frac{3}{2}\frac{1}{x}\frac{1}{c^x} + \int_{\delta}^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt \qquad \frac{1}{x}\frac{1}{c^x} \to 0$$

$$\leq \underbrace{\frac{3}{2}\frac{\delta^x}{x} + \int_0^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt}_{\text{an upper bound for } F(c)} \leftarrow \text{indep of } c$$

$\Rightarrow$ F is increasing and bounded above

$\Rightarrow$ $\displaystyle\int_0^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt = \lim_{c\to+\infty} F(c)$ exists (converges).

Convergence of (II):

$du = -dt$

$u = 1-t$

$$\int_{\frac{1}{2}}^{1} t^{x-1}(1-t)^{y-1}dt = -\int_{\frac{1}{2}}^{0} (1-u)^{x-1}u^{y-1}du$$

$$= \int_0^{\frac{1}{2}} u^{y-1}(1-u)^{x-1}du \leftarrow \text{same as (I)}$$

Above arguement $\Rightarrow$ $\displaystyle\int_{\frac{1}{2}}^{1} t^{x-1}(1-t)^{y-1}dt$ Converges

Conclusion:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad \text{Converges.} \quad \#$$

<u>Prop</u>

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}d\theta$$

pf

Let $t = \sin^2\theta$ (or $\theta = \sin^{-1}\sqrt{t}$)

$\Rightarrow dt = 2\sin\theta\cos\theta\, d\theta$

$\Rightarrow B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$

$$= \int_{0 = \sin^{-1}\sqrt{0}}^{\sin^{-1}\sqrt{1} = \frac{\pi}{2}} (\sin^2\theta)^{x-1} (\underbrace{1-\sin^2\theta}_{=\cos^2\theta})^{y-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2x-2+1}\theta\, \cos^{2y-2+1}\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\, \cos^{2y-1}\theta\, d\theta \quad \#$$

Thm

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $x, y > 0$.

pf: skip

Example

① $\Gamma(\frac{1}{2}) = ?$ $\left(= \int_0^\infty e^{-t} t^{\frac{1}{2}-1}\,dt\right)$

sol

$\int_0^\infty e^{-t}\, \cancel{t^{1-1}}\, dt = 1$

Since

$$B(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \frac{(\Gamma(\frac{1}{2}))^2}{\Gamma(1)}$$

$\|$

$\Big\|$ Prop

$$2\int_0^{\frac{\pi}{2}} (\sin\theta)^{\underbrace{2(\frac{1}{2})-1}_{=0}} \cdot (\cos\theta)^{\underbrace{2\cdot(\frac{1}{2})-1}_{=0}} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} 1 \, d\theta = 2\cdot\frac{\pi}{2} = \pi$$

we have

$$\left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = \pi$$

$$\Rightarrow \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \qquad \#$$

② $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = ?$

<u>sol</u>

Let $t = x^2 \Rightarrow dt = 2x\,dx$

$$\Rightarrow dx = \frac{1}{2\sqrt{t}} dt = \frac{1}{2} t^{-\frac{1}{2}} dt$$

even function

So $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-x^2} dx$

$$= \cancel{2}\int_0^{\infty} e^{-t} \cancel{\tfrac{1}{2}} t^{\overbrace{-\frac{1}{2}}^{=\frac{1}{2}-1}} dt$$

$$= \int_0^{\infty} e^{-t} t^{\frac{1}{2}-1} dt = \Gamma\left(\tfrac{1}{2}\right)$$

$$= \sqrt{\pi} \qquad \#$$

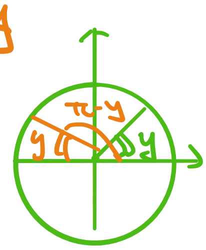
③ $\displaystyle\int_0^{2\pi} \sin^{100} x \, dx = ?$

Sol

Note that

$$\int_0^{2\pi} \sin^{100} x \, dx = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} + \int_{\pi}^{\frac{3}{2}\pi} + \int_{\frac{3}{2}\pi}^{2\pi}$$

$$\int_{\frac{\pi}{2}}^{\pi} \sin^{100} x \, dx \underset{dy=-dx}{\overset{y=\pi-x}{=}} \int_{\frac{\pi}{2}}^{0} (\underbrace{\sin(\pi-y)}_{=\sin y})^{100} (-dy)$$



$$= \int_0^{\frac{\pi}{2}} (\sin y)^{100} \, dy$$

Similarly,

$$\int_{\pi}^{\frac{3}{2}\pi} \sin^{100} x \, dx = \int_{\frac{3}{2}\pi}^{2\pi} \sin^{100} x \, dx = \int_0^{\frac{\pi}{2}} \sin^{100} x \, dx$$

So

$$\int_0^{2\pi} \sin^{100} x \, dx = 4 \int_0^{\frac{\pi}{2}} \sin^{100} x \, dx$$

$$= 2 \cdot 2 \int_0^{\frac{\pi}{2}} (\sin x)^{2 \cdot \frac{101}{2} - 1} \cdot (\cos x)^{2 \cdot \frac{1}{2} - 1} \, dx$$

$$= 2B\left(\frac{101}{2}, \frac{1}{2}\right)$$

$$= 2 \frac{\Gamma\left(\frac{101}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{101}{2} + \frac{1}{2}\right)} = 2 \frac{\Gamma\left(50 + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\underbrace{\Gamma(51)}_{= 50!}}$$

and $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma\left(50 + \frac{1}{2}\right) = \frac{99}{2} \Gamma\left(\frac{99}{2}\right) = \frac{99}{2} \cdot \frac{97}{2} \Gamma\left(\frac{97}{2}\right)$$

$$= \cdots = \frac{99}{2} \cdot \frac{97}{2} \cdot \ldots \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

So $\displaystyle\int_0^{2\pi} \sin^{100}x \, dx = 2B\left(\frac{101}{2}, \frac{1}{2}\right)$

$$= \frac{\frac{99}{2}\cdot\frac{97}{2}\cdots\frac{1}{2}\,\Gamma(\tfrac{1}{2})\cdot\Gamma(\tfrac{1}{2})}{50!} \qquad \Gamma(\tfrac{1}{2})\cdot\Gamma(\tfrac{1}{2}) = \pi$$

$$\frac{\overbrace{99\cdot 97\cdots 1\cdot(2\cdot 4\cdot 6\cdots 100)}^{100!}}{2^{49}\,\underbrace{(2\cdot 4\cdot 6\cdots 100)}_{2^{50}\cdot 50!}}$$

$$= \frac{100!}{2^{99}\,(50!)^2}\,\pi \qquad \#$$

④ $\displaystyle\int_2^5 (5-t)^{50}(t-2)^{50}\,dt = ?$

<u>sol</u>

Let $u = \dfrac{t-2}{3} \Rightarrow$

$$(5-t) = 5-(3u+2) = 3-3u = 3(1-u)$$

$$(t-2) = 3u$$

$$du = \frac{1}{3}dt$$

$$\Rightarrow \int_2^5 (5-t)^{50}(t-2)^{50}\,dt$$

$$= \int_{u(2)=0}^{u(5)=1} (3(1-u))^{50}\,(3u)^{50}\,3\,du$$

$$= 3^{101}\int_0^1 u^{51-1}(1-u)^{51-1}\,du$$

$$= 3^{101}\,B(51,51) = 3^{101}\cdot\frac{\Gamma(51)\Gamma(51)}{\Gamma(51+51)}$$

$$= 3^{101}\cdot\frac{50!\cdot 50!}{101!} \qquad \#$$

⑤ $\int_a^b (b-t)^{x-1} (t-a)^{y-1}\, dt = ?$

Sol

Let $u = \dfrac{t-a}{b-a} \Rightarrow du = \dfrac{1}{b-a} dt$

$t = (b-a)u + a$

$(b-t) = b-a-(b-a)u$
$= (b-a)(1-u)$

$(t-a) = (b-a)\cdot u$

$$\Rightarrow \int_a^b (b-t)^{x-1} (t-a)^{y-1}\, dt$$

$$= \int_0^1 ((b-a)(1-u))^{x-1} ((b-a)\cdot u)^{y-1} (b-a)\, du$$

$$= (b-a)^{x+y-1} \int_0^1 u^{y-1} (1-u)^{x-1}\, du$$

$$= (b-a)^{x+y-1} B(y,x) = \frac{\Gamma(y)\Gamma(x)}{\Gamma(x+y)} = B(x,y)$$

$$= (b-a)^{x+y-1} B(x,y)$$

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