Calculus, Spring 2026, week 7

Recall

We considered

$$V = \{ \text{continuous periodic functions with period } 2\pi \}$$

with $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$

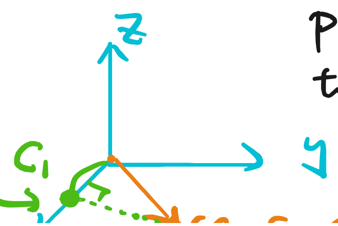
$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx$$

$$\| f \| := \sqrt{\langle f, f \rangle}$$

Then

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}} \right\} \cup \{ \cos mx, \sin nx \mid m, n \in \mathbb{N} \}$$

is "almost" an orthonormal basis for V.

$\mathcal{B}$	$\vec{i} = e_1$, $\vec{j} = e_2$, $\vec{k} = e_3$ $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ $(\mathbb{R}^3)$ standard o.n. basis
$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$ $\frac{a_0}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{a_0}{2}$ $a_k = \langle f(x), \cos kx \rangle$ $\left(\frac{a_0}{\sqrt{2}}, a_k, b_k \right)_{k=1}^{\infty}$	Coordinate $c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k} = (c_1, c_2, c_3) = \vec{v}$ $c_1 = \vec{v} \bullet \vec{i}$ $\quad c_3 = \vec{v} \bullet \vec{k}$ $c_2 = \vec{v} \bullet \vec{j}$
Best Approximation Thm: $\left\| \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx) - f(x) \right\|$ $= \min \ \| A_N(x) - f(x) \|$	$c_1 \vec{i} = (c_1, 0, 0) =$ the closest point in x-axis to (c_1, c_2, c_3) 

$A_N(x) \in$
$\text{span}\{\frac{1}{\sqrt{2}}, \cos nx, \sin mx \mid n, m = 1, \dots, N\}$

$$\|f(x)\| = \sqrt{\left(\frac{a_0}{\sqrt{2}}\right)^2 + \sum_{k=1}^{\infty} a_k^2 + b_k^2}$$

Parseval's identity

$$\langle f(x), g(x) \rangle = \frac{a_0}{\sqrt{2}} \cdot \frac{\alpha_0}{\sqrt{2}} + \sum_{k=1}^{\infty} (a_k \cdot \alpha_k + b_k \cdot \beta_k)$$

where

$$g(x) \sim \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos kx + \beta_k \sin kx$$

$$\|(c_1, c_2, c_3)\| = \sqrt{c_1^2 + c_2^2 + c_3^2}$$

$$(c_1, c_2, c_3) \bullet (d_1, d_2, d_3) = c_1 d_1 + c_2 d_2 + c_3 d_3$$

Thm (Parseval's identity)

Let $f(x)$ be a piecewise continuous periodic function with period 2π. If

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

then

$$\|f\|^2 \overset{\text{def}}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx \overset{\text{thm}}{=} \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

pf: skip.

Cor

Let f, g be piecewise continuous 2π-periodic functions. If

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

$$g(x) \sim \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx).$$

Then

$$\langle f,g\rangle \overset{\text{def}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx \overset{\text{thm}}{=} \frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty}(a_k\alpha_k + b_k\beta_k).$$

pf.

Note that

$$\cancel{\langle f,f\rangle} + \underline{2\langle f,g\rangle} + \cancel{\langle g,g\rangle} \qquad \cancel{\langle f,f\rangle} - 2\langle f,g\rangle \cancel{+\langle g,g\rangle}$$

$$\frac{\|f+g\|^2 - \|f-g\|^2}{4} = \frac{\langle f+g, f+g\rangle - \langle f-g, f-g\rangle}{4}$$

$$= \langle f, g\rangle$$

$$\Rightarrow \langle f,g\rangle = \frac{1}{4}\left(\left(\frac{(a_0+\alpha_0)^2}{2} + \sum_{k=1}^{\infty}\left((a_k+\alpha_k)^2 + (b_k+\beta_k)^2\right)\right) - \left(\frac{(a_0-\alpha_0)^2}{2} + \sum_{k=1}^{\infty}\left((a_k-\alpha_k)^2 + (b_k-\beta_k)^2\right)\right)\right)$$

Parseval's identity

$$= \frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty}(a_k\alpha_k + b_k\beta_k) \qquad \#$$

Example

Prove $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$ by Parseval's identity pf

Recall that if $f(x)$ is the 2π-periodic function s.t.

$$f(x) = x, \qquad -\pi < x \le \pi,$$

then

$$f(x) \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx.$$

By Parseval's identity,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=1}^{\infty} \left(\frac{2(-1)^{k+1}}{k}\right)^2 = \sum_{k=1}^{\infty} \frac{4}{k^2}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = 2\frac{1}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4}\cdot\frac{2\pi^2}{3} = \frac{\pi^2}{6} \qquad \#$$

§ Special functions

Legendre polynomials

idea:

Try to make $1, x, x^2, x^3, \cdots$ orthogonal and get properties similar to Fourier series

More precisely, consider:

$V = C[-1,1] = \{\text{continuous functions on } [-1,1]\}$

and

$\mathbb{R}[x] = \{\text{polynomials with real coefficients}\}$

Define

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)\,dx$$

which is an inner product on $V \supseteq \mathbb{R}[x]$

Note that the standard basis

$$\overset{v_0}{\overset{\shortparallel}{1}},\ \overset{v_1}{\overset{\shortparallel}{x}},\ \overset{v_2}{\overset{\shortparallel}{x^2}},\ \overset{v_3}{\overset{\shortparallel}{x^3}},\ \ldots,\ \overset{v_n}{\overset{\shortparallel}{x^n}},\ \ldots$$

for $\mathbb{R}[x]$ is NOT orthogonal:

$$\langle 1, x^2 \rangle = \int_{-1}^{1} 1 \cdot x^2\,dx$$

$$= \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} \neq 0$$

To obtain an orthogonal basis for $\mathbb{R}[x]$ the standard method is "Gram-Schmidt process" $\{v_n\} \xrightarrow{\text{G-S}} \overset{\text{o.g.}}{\{u_n\}} \xrightarrow{\text{normalization}} \{P_n(x)\}$

Here, by convention, we impose the normalization condition: $P_n(1) = 1$

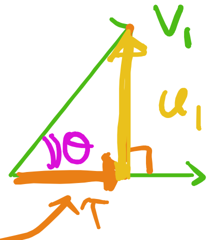
(i) For $n=0$, let $v_0 = 1$ ($u_0 = v_0 \Rightarrow$), $\because v_0(1) = 1$, set $P_0(x) = 1$

(ii) For $n=1$, let

$\|v_1\|\cdot\|v_0\|\cdot\cos\theta$

$$u_1 = v_1 - \frac{\langle v_1 , v_0 \rangle}{\langle v_0 , v_0 \rangle} v_0$$

$\|v_0\|\cdot\|v_0\|$



norm $= \|V_1\|\cdot\cos\theta$

" $\dfrac{\|V_1\|\cos\theta}{\|V_0\|}$ "

$$= x - \frac{\langle x , 1 \rangle}{\langle 1 , 1 \rangle} 1$$

$$= x - \frac{\int_{-1}^{1} x \cdot 1 \, dx}{\int_{-1}^{1} 1 \cdot 1 \, dx} \cdot 1 = x$$

Since $u_1(1) = 1$, we set $P_1(x) = u_1(x) = x$

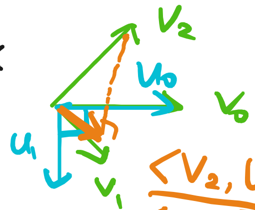
(iii) For $n=2$,

$$u_2 = v_2 - \frac{\langle v_2 , u_0 \rangle}{\langle u_0 , u_0 \rangle} u_0 - \frac{\langle v_2 , u_1 \rangle}{\langle u_1 , u_1 \rangle} u_1$$

$$= x^2 - \frac{\int_{-1}^{1} x^2 \cdot 1 \, dx}{\int_{-1}^{1} 1 \cdot 1 \, dx} \cdot 1 - \frac{\int_{-1}^{1} x^2 \cdot x \, dx}{\int_{-1}^{1} x \cdot x \, dx} x$$

$= \frac{2}{3}$

2



$$\frac{\langle V_2, U_0\rangle}{\langle U_0, U_0\rangle} U_0 + \frac{\langle V_2, U_1\rangle}{\langle U_1, U_1\rangle} U_1$$

$$= x^2 - \frac{1}{3}$$

$$\Rightarrow P_2(x) = \frac{u_2(x)}{u_2(1)} = \frac{x^2 - \frac{1}{3}}{1 - \frac{1}{3}}$$

$$= \frac{1}{2}(3x^2 - 1)$$

(iv) For $n=3$,

$$u_3 = v_3 - \frac{\langle v_3 , u_0 \rangle}{\langle u_0 , u_0 \rangle} u_0 - \frac{\langle v_3 , u_1 \rangle}{\langle u_1 , u_1 \rangle} u_1 - \frac{\langle v_3 , u_2 \rangle}{\langle u_2 , u_2 \rangle} u_2$$

$$= \cdots = x^3 - \frac{3}{5} x$$

$$\Rightarrow \quad P_3(x) = \frac{u_3(x)}{u_3(1)} = \frac{1}{2}(5x^3 - 3x)$$

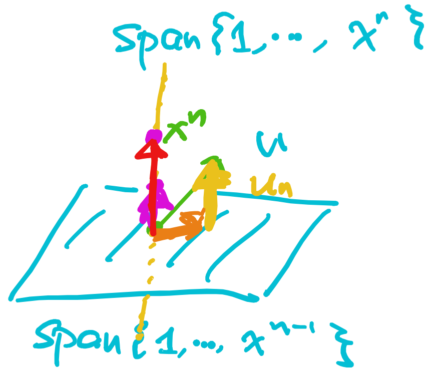
$\vdots$

By continuing this inductive process, one can obtain all the $P_n(x)$, which are called the Legendre polynomials.

Prop (Rodrigues' formula)

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

pf



Let $R_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

Note that

$$\mathrm{span}\{1, x, \dots, x^n\} = \mathrm{span}\{P_0, P_1, \cdots, P_n\}$$

and

$P_n(x) =$ the unique vector which is ① orthogonal to $\mathrm{span}\{1, x, \dots, x^{n-1}\} = \mathrm{span}\{P_0, P_1, \dots, P_{n-1}\}$

and ② $P_n(1) = 1$

So it suffices to show that $R_n(x)$ satisfies ① and ②

$\int v'u\,dx = vu - \int u'v\,dx$

① $$\langle R_n, x^m \rangle \overset{m<n}{=} \frac{1}{2^n n!} \int_{-1}^{1} \Big(\underbrace{\frac{d^n}{dx^n}(x^2-1)^n}_{v'(x)} \Big) \cdot \underbrace{x^m}_{u(x)}\, dx$$

$$= \frac{1}{2^n n!}\left(\cancel{\frac{d^{n-1}}{dx^{n-1}}((x^2-1)^n)}^{=0} x^m \Big|_{-1}^{1} - \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \cdot m x^{m-1} dx \right)$$

$$\frac{d}{dx}\left((x^2-1)^n\right) = n(x^2-1)^{n-1} \cdot 2x$$

$$\frac{d^2}{dx^2}(x^2-1)^n = n(n-1)(x^2-1)^{n-2} \cdot 2x + n(x^2-1)^{n-1} \cdot 2$$

$$= (x^2-1)^{n-2}(\cdots\cdots)$$

$$\frac{d^3}{dx^3}(x^2-1)^n = (x^2-1)^{n-3}(\cdots\cdots)$$

$$\vdots$$

$$\frac{d^k}{dx^k}(x^2-1)^n \Big|_{x=\pm 1} = (x^2-1)^{n-k} \cdot (\cdots\cdots)\Big|_{x=\pm1} \overset{\text{if } k<n}{=0}$$

$$= \frac{-1}{2^n n!} \int_{-1}^{1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \cdot m x^{m-1} dx$$

$$= \frac{\pm 1}{2^n n!} \int_{-1}^{1} \frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n \, m(m-1) x^{m-2} dx$$

$$= \cdots = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} (x^2-1)^n \frac{d^n}{dx^n}(x^m) dx = 0$$

② $R_n(1) = ?$, $R_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

Recall

$$\frac{d^n}{dx^n}(f(x)g(x)) = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{= C^n_k} f^{(k)}(x)\, g^{(n-k)}(x)$$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$(f(x)g(x))'' = f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x)$$

$$= f''(x) + 2f'(x)g'(x) + g''(x)$$

$$\frac{d^n}{dx^n}(x^2-1)^n\Big|_{x=1} = \frac{d^n}{dx^n}\left((x+1)^n\cdot(x-1)^n\right)\Big|_{x=1}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}(x+1)^n\Big|_{x=1}\cdot\underbrace{\frac{d^{n-k}}{dx^{n-k}}(x-1)^n\Big|_{x=1}}_{=\,0 \text{ if } n-k<n}$$

Recall

If $f(c)=0$, and $p<n$, then

$$\Rightarrow \underbrace{\frac{d^p}{dx^p}(f(x))^n}_{\text{(induction)}}\Big|_{x=c} = 0$$

$$f(x)\cdot(\cdots\cdots)\Big|_{x=c} = 0$$

$$= \binom{n}{0}\underbrace{(x+1)^n\Big|_{x=1}}_{=\,2^n}\cdot\underbrace{\frac{d^n}{dx^n}(x-1)^n\Big|_{x=1}}_{=\,\frac{d^{n-1}}{dx^{n-1}}\left(n(x-1)^{n-1}\right)=\cdots}$$

$$n(n-1)(n-2)\cdots 1 = n!$$

$$= 2^n\cdot n!$$

$$\Rightarrow P_n(1) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n\Big|_{x=1}$$

$$= \frac{1}{2^n n!}\cdot 2^n n! = 1 \quad \#$$

c.f. $\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \cdots$ in Fourier series

Thm

$$\langle P_n, P_m\rangle = \int_{-1}^{1}P_n(x)P_m(x)\,dx$$

$$= \begin{cases} 0, & n \neq m \\ \dfrac{2}{2n+1} & n = m \end{cases}$$

pf

It remains to show that $\langle P_n, P_n \rangle = \dfrac{2}{2n+1}$:

$\int u'v\,dx = uv - \int v'u\,dx$

$$\langle P_n, P_n \rangle = \int_{-1}^{1} P_n(x) P_n(x)\,dx$$

$$= \int_{-1}^{1} \underbrace{\frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n}_{u'} \cdot \underbrace{\frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n}_{v}\,dx$$

$$= \left(\frac{1}{2^n n!}\right)^2 \cdot \Bigg(\underbrace{\cancel{\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n}}_{=0 \text{ at } x=\pm 1} \cdot \frac{d^n}{dx^n}(x^2-1)^n \Bigg|_{-1}^{1} = 0$$

$n-1 < n$

$$- \int_{-1}^{1} \underbrace{\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n}_{u'} \cdot \underbrace{\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n}_{v}\,dx \Bigg)$$

$$= \left(\frac{1}{2^n n!}\right)^2 (-1)^2 \cdot \int_{-1}^{1} \frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n \cdot \frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n\,dx$$

$$= \cdots$$

polynomial of deg $2n$

$$= \left(\frac{1}{2^n n!}\right)^2 (-1)^n \int_{-1}^{1} (x^2-1)^n \cdot \frac{d^{2n}}{dx^{2n}} \boxed{(x^2-1)^n}\,dx$$

$$\frac{d^{2n}}{dx^{2n}}\left(x^{2n} + \boxed{\binom{n}{1} x^{2n-2} + \cdots}\right) = (2n)!$$

$$= \frac{(-1)^n (2n)!}{(2^n n!)^2} \int_{-1}^{1} (x^2-1)^n \, dx$$

$$= \frac{(2n)!}{(2^n n!)^2} \int_{-1}^{1} (1-x^2)^n \, dx \; = ?$$

Let $x = \cos\theta$ (or $\theta = \cos^{-1} x$)

$dx = -\sin\theta \, d\theta$

$x = 1 \leftrightarrow \theta = 0$

$x = -1 \leftrightarrow \theta = \pi$

$$\Rightarrow \int_{-1}^{1} (1-x^2)^n dx = \int_{\pi}^{0} \underbrace{(1-\cos^2\theta)^n}_{\sin^{2n}\theta} (-\sin\theta) \, d\theta$$

see Remark below

$$= \int_0^{\pi} \sin^{2n+1}\theta \, d\theta \; \text{"}= B(n+1, \tfrac{1}{2})\text{"}$$

$2x-1$

Beta function HW8

$$= \frac{2^{2n+1} (n!)^2}{(2n+1)!}$$

$$\Rightarrow \langle P_n, P_n \rangle = \frac{\cancel{(2n)!}}{(\cancel{2^n} \cancel{n!})^2} \cdot \frac{\cancel{2^{2n+1}}^{\,2} \, \cancel{(n!)^2}}{\cancel{(2n+1)!}_{\,2n+1}} = \frac{2}{2n+1}$$

\#

Similarly to Fourier series, we have:

Thm (Best Polynomial Approximation)

Given $f \in C[-1,1]$, consider

$$A_N(x) = \sum_{n=0}^{N} a_n \cdot P_n(x) \quad \leftarrow \text{poly of deg } N$$

and

$$\| f(x) - A_N(x) \|^2 = \int_{-1}^{1} (f(x) - A_N(x))^2 \, dx$$

Let

$$S_N(x) = \sum_{n=0}^{N} C_n P_n(x) \leftarrow \quad \text{c.f.} \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

where

$$C_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \, dx$$

Then

$$\| f(x) - A_N(x) \|^2 \geq \| f(x) - S_N(x) \|^2$$

and "=" holds $\Leftrightarrow$ $A_N(x) = S_N(x)$.

Remark

$$\int u' v \, dx = uv - \int u v' \, dx$$

$$\int_{-1}^{1} (1-x^2)^n \, dx = \int_{-1}^{1} \underbrace{(1-x)^n}_{u'} \underbrace{(1+x)^n}_{v} \, dx$$

$$= \underbrace{- \frac{(1-x)^{n+1}}{n+1} (1+x)^n \Big|_{-1}^{1}}_{=0} + \int_{-1}^{1} \underbrace{\frac{(1-x)^{n+1}}{n+1}}_{u} \underbrace{n(1+x)^{n-1}}_{v'} \, dx$$

$$= \frac{n}{n+1} \left(\underbrace{- \frac{(1-x)^{n+2}}{n+2} (1+x)^{n-1} \Big|_{-1}^{1}}_{=0} + \int_{-1}^{1} \frac{(1-x)^{n+2}}{n+2} (n-1)(1+x)^{n-2} \, dx \right)$$

$$= \cdots$$

$$= \frac{n}{n+1} \cdot \frac{n-1}{n+2} \cdots \frac{1}{n+n} \underbrace{\int_{-1}^{1} (1-x)^{2n} \, dx}$$

$$- \frac{0 - 2^{2n+1}}{2n+1} = - \frac{(1-x)^{2n+1}}{2n+1} \Big|_{-1}^{1}$$

$$= \frac{1}{(n+1)(n+2)\cdots 2n} \cdot \frac{2^{2n+1}}{2n+1} = \frac{n!}{(2n+1)!}$$

$$= \frac{(n!)^2 \cdot 2^{2n+1}}{(2n+1)!} \qquad \#$$

<u>sketch of pf</u> (Best Polynomial Approximation)

$$\|f - A_N\|^2 = \langle f - A_N, f - A_N \rangle$$

$$= \|f\|^2 - 2\underline{\langle f, A_N \rangle} + \underline{\|A_N\|^2}$$

$$\langle f, A_N \rangle = \langle f, \sum_{n=0}^{N} a_n P_n \rangle = \sum_{n=0}^{N} a_n \langle f, P_n \rangle$$

$$= \sum_{n=0}^{N} a_n \cdot c_n \cdot \langle P_n, P_n \rangle \quad \left(c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} \right)$$

$$\|A_N\|^2 = \langle \sum_{n=0}^{N} a_n P_n , \sum_{m=0}^{N} a_m P_m \rangle$$

$$= \sum_{n,m=0}^{N} a_n a_m \langle P_n, P_m \rangle = \begin{cases} 0 & n \neq m \\ \frac{2}{2n+1} = \langle P_n, P_n \rangle \end{cases}$$

$$= \sum_{n=0}^{N} a_n^2 \langle P_n, P_n \rangle$$

$$\Rightarrow \|f - A_N\|^2 = \|f\|^2 - \sum_{n=0}^{N} \underline{2a_n c_n} \langle P_n, P_n \rangle$$

$$+ \sum_{n=0}^{N} \underline{a_n^2} \langle P_n, P_n \rangle$$

$$+ \sum_{n=0}^{N} (\underline{c_n^2} - c_n^2) \langle P_n, P_n \rangle$$

$$= \|f\|^2 + \sum_{n=0}^{N} (a_n - c_n)^2 \underbrace{\langle P_n, P_n \rangle}_{} = \frac{2}{2n+1}$$

$$- \sum_{n=0}^{N} c_n^2 \langle P_n, P_n \rangle$$

$$\geq \|f\|^2 - \sum_{n=0}^{N} c_n^2 \langle P_n, P_n \rangle$$

$$= \|f - S_N\|^2$$

"=" holds $\iff$ $a_n = c_n \ \forall n = 0, 1, \cdots, N$. #

2. Find the interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{1}{k2^k} x^k$.

$x = -2$: $\sum_{k=1}^{\infty} \frac{1}{k2^k}(-2)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

易錯：

- 注意到在說明交錯級數 $\sum_{k=1}^{\infty}(-1)^k b_k$ 收斂時 b_k 必須要單調遞減收斂到 0，單純收斂到 0 並不保證。

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3. Sum the series $\sum_{k=1}^{\infty} \frac{k(k+1)}{3^k}$.

$\left(\sum_{k=0}^{\infty} x^k\right)' = \left(\frac{1}{1-x}\right)'$ $\forall$ $x \in (-1,1)$

易錯：

- 本題應該指出 $p(x) = \sum_{k=1}^{\infty} k(k+1)x^k$ 的收斂半徑是 1，首先大部分同學都採用微分的做法，這最好要標明有效的範圍。更重要的是因為最終我們帶入 $x = \frac{1}{3}$ 必須在區間內，最常見的反例是 $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$，若 $x = 2$ 帶入左邊得 -1 但右邊發散。

Average	34					
Average (except zero)	38.95145631					
Quartile 0	0					
Quartile 1	20					
Quartile 2	38					
Quartile 3	53					
Quartile 4	60					
標準差	19.72719932					
非零數	103					
不到50%人數	43					
滿分人數	5	0	0	0	0	0