Calculus, Spring 2026, week 6

Recall

Let $f(x) \in PC^1$, 2π-periodic function

If

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \frac{f(x_0^+) + f(x_0^-)}{2}$$

Here,

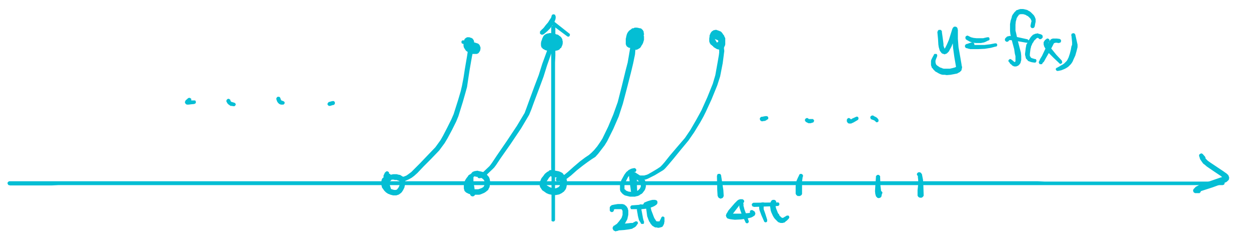
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

Application to Computing $\sum_{k=1}^{\infty} \frac{1}{k^2}$:

Consider the 2π-periodic function $f(x)$ satisfying

$$f(x) = x^2, \qquad \forall x \in (0, 2\pi]$$



The Fourier series of $f(x)$ is ...

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cdot\cos kx}_{2\pi\text{-periodic}}\,dx$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos kx\,dx$$

$$\left(\frac{\sin kx}{k}\right)' \qquad \left(\frac{-\cos kx}{k}\right)'$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2 \cos kx\,dx$$

if $k\neq 0$

integration by parts

$$= \frac{1}{\pi}\left(\underbrace{x^2\cdot\frac{\sin kx}{k}\Big|_0^{2\pi}}_{=0} - \int_0^{2\pi} 2x\,\frac{\sin kx}{k}\,dx \right)$$

$$= \frac{1}{\pi}\left(-2x\cdot\frac{(-\cos kx)}{k^2}\Big|_0^{2\pi} - \underbrace{\int_0^{2\pi}\frac{2}{k^2}\cos kx\,dx}_{=0} \right)$$

$$= \frac{-1}{\pi}\cdot 4\pi\cdot\frac{-1}{k^2} = \frac{4}{k^2}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\cdot\frac{8\pi^3}{3} = \frac{8\pi^2}{3}$$

$$\left(\frac{x^3}{3}\right)'$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin kx\,dx = \cdots\cdots$$

$$= \frac{-4\pi}{k}$$

$$\Rightarrow f(x) \sim \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty}\left(\frac{\cos kx}{k^2} - \pi\,\frac{\sin kx}{k}\right)$$

At $x = 0$

$$\frac{f(0^+)+f(0^-)}{2} = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)$$

$$\| \qquad\qquad\qquad\qquad //$$

$$\frac{0 + 4\pi^2}{2} = 2\pi^2$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{4}\Bigg(\overset{\frac{2}{3}\pi^2}{\cancel{2\pi^2 - \frac{4\pi^2}{3}}}\Bigg) = \frac{\pi^2}{6}$$

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Linear alg perspective

Let

$$V = \{\text{ continuous differentiable } 2\pi\text{-periodic functions }\}$$

which is a vector space over $\mathbb{R}$:

for $r \in \mathbb{R}$, $f, g \in V$,

$$(r\cdot f)(x) = r\cdot f(x)$$

$$(f+g)(x) = f(x) + g(x)$$

Define

$$\langle\, ,\rangle : V \times V \to \mathbb{R}$$

by

$$\langle f, g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot g(x)\,dx$$

inner product

and

$$\|f\| := \sqrt{\langle f, f\rangle}$$

← norm

<u>Lemma</u>

$(V, \langle\ ,\ \rangle)$ is an inner product space

That is, $\forall f, g, h \in V$, $r \in \mathbb{R}$,

(i) $\langle r\cdot f + g\ ,\ h\rangle =$

$r\langle f, h\rangle + \langle g, h\rangle$

(ii) $\langle f, g\rangle = \langle g, f\rangle$

(iii) $\langle f, f\rangle \geq 0$ and

$\langle f, f\rangle = 0 \Leftrightarrow f = 0$ in V

$f(x) = 0 \ \forall x \in \mathbb{R}$

<u>Remark</u>:

If $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

then $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot f(x)\,dx = 0$

But $f(x) \notin V$.

Def

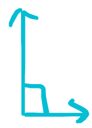
A subset $S \subseteq V$ is orthogonal if $0 \notin S$, and

$$\langle f, g \rangle = 0 \quad \forall\ f \neq g \in S.$$

We say $S \subseteq V$ is orthonormal if S is orthogonal and

$$\|f\| = 1 \quad \forall f \in S.$$

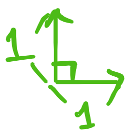
Remark



If $S \subseteq V$ is orthogonal, then

$$S' := \left\{ \frac{f}{\|f\|} \;\middle|\; f \in S \right\}$$

is orthonormal, and



$$\operatorname{span} S = \operatorname{span} S'$$

Lemma

Any orthogonal set is linearly independent.

Let

$$S = \{\underset{\cos 0x}{1}\} \cup \{\cos mx, \sin nx \mid n, m \in \mathbb{N}\}$$

Since

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \langle \cos mx, \cos nx \rangle \begin{cases} = 0 & \text{if } n \neq m \\ = \frac{1}{2} & \text{if } n = m \neq 0 \\ = \text{ } & \text{if } n = m = 0 \end{cases}$$

$\underline{\qquad}$ 1

and

$$\langle \cos mx, \sin nx \rangle = 0$$

$$\langle \sin mx, \sin nx \rangle \begin{cases} = 0 & \forall\, n \neq m \\ = 1 & n = m \end{cases}$$

we know that S is orthogonal in $(V, \langle , \rangle)$, and

$$S' = \left\{\frac{1}{\sqrt{2}}\right\} \cup \{\cos mx, \sin nx \mid m, n \in \mathbb{N}\}$$

is orthonormal in $(V, \langle , \rangle)$

$\Rightarrow$ S (or S') is a basis for

$$W := \operatorname{span} S$$

$$= \left\{ \underbrace{\frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx)}_{\text{smooth} = \text{differentiable } \infty \text{ times}} \;\middle|\; N \in \mathbb{N},\ a_k, b_k \in \mathbb{R} \right\}$$

Note that W is strictly smaller than V

$$W \subsetneq V$$

$=$ $\leftarrow$ continuous function

 $\in V \setminus W$

But by Pointwise Conv. Thm, any $f \in V$ can be represented as the limit of a sequence in W.

The Fourier series of $f(x) \in V$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

can be considered as "the coordinate of $f(x)$" with respect to S

$$a_k = \langle f(x), \cos kx \rangle$$

$$b_k = \langle f(x), \sin kx \rangle$$

$$= \left\langle \frac{a_0}{2}, \cos kx \right\rangle + \sum_{n=1}^{N \to \infty} a_n \langle \cos nx, \cos kx \rangle + b_n \langle \sin nx, \cos kx \rangle$$

$$= a_k \cdot \langle \cos kx, \cos kx \rangle = a_k$$

Let

$$W_N = \operatorname{span}\{1, \cos x, \sin x, \dots, \cos Nx, \sin Nx\}$$

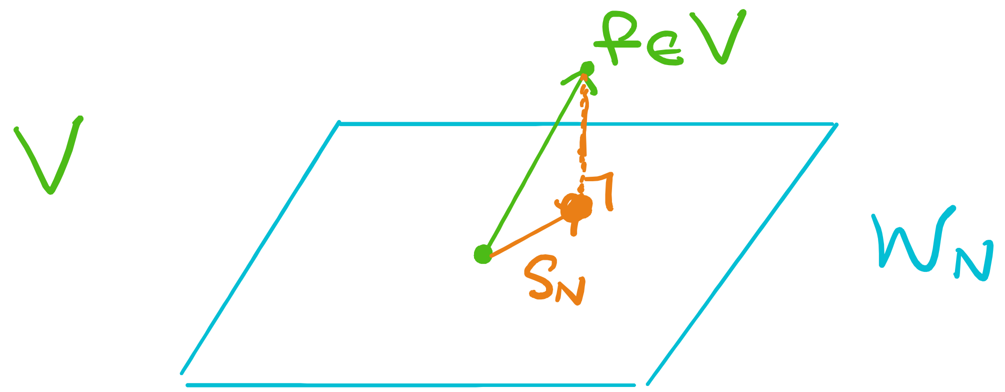
$$= \left\{ \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx) \right\}$$

Let's try to find a best approximation in W_N of $f \in V$.

Here, by "best", we mean S_N minimizes $\|f - A_N\|$, $A_N \in W_N$.

That is,

$$\|f - S_N\| \leq \|f - A_N\| \quad \forall A_N \in W_N$$



<u>Thm</u> (Best Approximation Thm)

Let f be a piecewise continuous 2π-periodic function, and let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

and

$$S_N(x) := \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx)$$

Then $\forall\ \alpha_0, \alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_n \in \mathbb{R}$, consider

$$A_N(x) := \frac{\alpha_0}{2} + \sum_{k=1}^{N} (\alpha_k \cos kx + \beta_k \sin kx)$$

and we have

$$\|f(x) - S_N(x)\| \leq \|f(x) - A_N(x)\|$$

Furthermore, "=" holds $\Leftrightarrow$

$$\alpha_k = a_k \quad \forall k=0,\dots,N$$
$$\beta_k = b_k \quad \forall k=1,\dots,N$$

Pf

$$\|f-A_N\|^2 = \langle f-A_N, f-A_N\rangle$$
$$= \underbrace{\|f\|^2}_{①} - 2\underbrace{\langle f, A_N\rangle}_{②} + \underbrace{\|A_N\|^2}_{③}$$

and

② $$\langle f, A_N\rangle = \langle f, \frac{\alpha_0}{2} + \sum_{k=1}^{N} \alpha_k \cos kx + \beta_k \sin kx\rangle$$
$$= \frac{\alpha_0}{2}\underbrace{\langle f, 1\rangle}_{a_0} + \sum_{k=1}^{N} \alpha_k \underbrace{\langle f, \cos kx\rangle}_{a_k} + \beta_k \underbrace{\langle f, \sin kx\rangle}_{b_k}$$
$$= \frac{\alpha_0 a_0}{2} + \sum_{k=1}^{N} (\alpha_k a_k + \beta_k b_k)$$

③ $$\|A_N\|^2 = \langle A_N, A_N\rangle$$
$$= \frac{\alpha_0^2}{2} + \sum_{k=1}^{N} (\alpha_k^2 + \beta_k^2)$$

$$= \langle \frac{\alpha_0}{2} + \sum_{k=1}^{N} \alpha_k \cos kx + \beta_k \sin kx, \quad \dots \rangle$$
$$= \cdots \alpha_0\alpha_0 \langle \tfrac{1}{2}, \tfrac{1}{2}\rangle + \alpha_i \alpha_j \cancel{\langle \cos ix, \cos jx\rangle} \quad = 0 \text{ if } i \neq j$$
$$+ \cdots$$
$$= \cdots$$

$$\Rightarrow \|f-A_N\|^2 = \|f\|^2 - \left(\alpha_0 a_0 + 2\sum_{k=1}^{N} (\alpha_k a_k + \beta_k b_k)\right)$$
$$+ \left(\frac{\alpha_0^2}{2} + \sum_{k=1}^{N} (\alpha_k^2 + \beta_k^2)\right) - (*) + (*)$$

If $A_N = S_N$,

$$\|f - S_N\|^2 = \|f\|^2 - \underbrace{\left(\frac{a_0^2}{2} + \sum_{k=1}^{N} (a_k^2 + b_k^2)\right)}_{(*)}$$

Note that

$$\|f-A_N\|^2 = \|f-S_N\|^2 + \left(\underbrace{\frac{(\alpha_0-a_0)^2}{2} + \sum_{k=1}^{N}\left((\alpha_k-a_k)^2+(\beta_k-b_k)^2\right)}_{\geq 0}\right)$$

$$\geq \|f-S_N\|^2$$

and "=" holds $\Leftrightarrow$ $\begin{cases} \alpha_k = a_k & k=0,1,\cdots,N \\ \beta_k = b_k & k=1,\cdots,N \end{cases}$

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