Calculus, Spring 2026, week 5

Summary of Taylor series and power series

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Smooth functions

$$
\begin{tikzcd}
f(x) \arrow[r, shift left, "\text{Taylor series}"] & \displaystyle\sum_{k=0}^{\infty} a_k x^k \arrow[l, shift left, "\text{Compute sum/limit}"]
\end{tikzcd}
$$

power series

- Taylor series (at 0):
  - $$f(x) \longrightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$
  - $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ does NOT necessarily converge to $f(x)$. For example, if

$$
f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}
$$

(HW in last semester)

then $f(x)$ is a smooth function and $f^{(k)}(0) = 0 \quad \forall k \geq 0$

$\Rightarrow$ Taylor series of $f(x)$ (at 0) $= 0$

which does NOT converge to $f(x)$

$\forall x \neq 0$.

- To get $f(x)$, we can consider the finite expansion:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some c between 0 and x.

- $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad (*)$$

$$\Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$$

- To get an equality like $(*)$, we usually try to get the desired expansion by the power series techniques and well-known expansions:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2!} + \cdots \quad \forall x \in \mathbb{R}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \forall x \in \mathbb{R}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad \forall x \in \mathbb{R}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \quad \forall x \in (-1, 1)$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} \quad \forall x \in (-1, 1]$$

- Power series : $\sum_{k=0}^{\infty} a_k x^k$
  - Many nice properties:
    radius of convergence, differentiation, integration, etc.
  - If $\sum_{k=0}^{\infty} a_k x^k$ Converges on $(-r, r)$, $r>0$, consider

$$f(x) \doteq \sum_{k=0}^{\infty} a_k x^k, \quad \forall x \in (-r, r)$$

$\Rightarrow$ Taylor series of $f(x)$ at 0 $= \sum_{k=0}^{\infty} a_k x^k$.

Fourier Series

Idea: we want to approximate $f(x)$ by $\sin(kx)$ and $\cos(kx)$:

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx)\right)$$

Note that

$$\cos k(x+2\pi) = \cos(kx+2k\pi) = \cos kx$$

$$\sin k(x+2\pi) = \sin kx$$

Def

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A function $f: \mathbb{R} \to \mathbb{R}$ is periodic with period $T > 0$ if

$$f(x+T) = f(x) \quad \forall x \in \mathbb{R}$$

For example, $\cos kx$, $\sin kx$ are periodic function with period 2π.

Lemma

If f is a continuous periodic function with period T

then
$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

pf

Let $F(a) = \int_a^{a+T} f(x)\,dx$

$$\Rightarrow \quad F'(a) = \left(\int_0^{a+T} f(x)\,dx - \int_0^a f(x)\,dx \right)'$$

$$= \underbrace{f(a+T)}_{f(a)} - f(a) = 0 \quad \forall a \in \mathbb{R}$$

$\Rightarrow \quad F(a) \equiv c$ for some $c \in \mathbb{R}$

In particular,

$$c = F(0) = \int_0^T f(x)\,dx = \int_a^{a+T} f(x)\,dx \quad \forall a \in \mathbb{R}$$

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Now consider a periodic function $f(x)$ with period 2π.

Hope:

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

Q: What should be a_k, b_k ?

To guess a_k, b_k, assume integration

of $f(x)$ can be computed by term-by-term integration. Then

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx$$

$$" = " \int_{-\pi}^{\pi} \left(\frac{a_0}{2} \cos mx + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \right) dx$$

$$" = " \underbrace{\int_{-\pi}^{\pi} \frac{a_0}{2} \cos mx \cdot \cos 0x \, dx}_{= \begin{cases} 0 & m \neq 0 \\ 2\pi & m = 0 \end{cases}}$$

$$+ \sum_{k=1}^{\infty} \Bigg(\underbrace{\int_{-\pi}^{\pi} a_k \cos kx \cdot \cos mx \, dx}_{= \begin{cases} 0 & k \neq m \\ a_k \pi & k = m \end{cases}} + \underbrace{\int_{-\pi}^{\pi} b_k \sin kx \cdot \cos mx \, dx}_{= 0} \Bigg)$$

Q: $\displaystyle \int_{-\pi}^{\pi} \cos kx \cos mx \, dx = ?$

$\displaystyle \int_{-\pi}^{\pi} \sin kx \cos mx \, dx = ?$

<u>Lemma</u>

For non-negative integers n, m, we have

(i) $$\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n \neq 0 \end{cases}$$

(ii) $$\int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ 0 & \text{if } m = n = 0 \\ \cancel{2\pi}\ \pi & \text{if } m = n \neq 0 \end{cases}$$

(iii) $$\int_{-\pi}^{\pi} \sin mx \cdot \cos nx \, dx = 0 \qquad \forall n, m \in \mathbb{N} \cup \{0\}$$

<u>pf of (i)</u>:

Recall
$$\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$\Rightarrow \quad \cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha \cos\beta$$

Take $\alpha = mx$, $\beta = nx$:

$$\cos mx \cos nx = \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right)$$

$$\Rightarrow \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x + \underline{\cos(m-n)x} \, dx$$

$$= \frac{1}{2}\left(\frac{\sin(m+n)x}{\underbrace{m+n}_{\neq 0}} + \frac{\sin(m-n)x}{\underbrace{m-n}_{\neq 0}}\right)\Bigg|_{x=-\pi}^{\pi}$$

if $m \neq n \geq 0$

$$= 0$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \underbrace{\cos 0x + \cos 0x}_{=2}\, dx = 2\pi$$

if $m=n=0$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \underbrace{\cos(m+n)x}_{=\left(\frac{\sin(m+n)x}{m+n}\right)'} + \underbrace{\cos 0x}_{=1}\, dx$$

if $m=n\neq 0$

$$= \frac{1}{2}\left(\underbrace{\cancel{\frac{\sin(m+n)x}{m+n}\Bigg|_{x=-\pi}^{\pi}}}_{=0} + 2\pi\right) = \pi$$

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By Lemma, we expect that

$$\int_{-\pi}^{\pi} f(x)\cdot \underset{\cos 0x\ (m=0)}{1}\, dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\cdot\cos(0x)\,dx = \frac{a_0}{2}\cdot 2\pi$$

$$= a_0\pi$$

$$\int_{-\pi}^{\pi} f(x) \underset{(m\neq 0)}{\cos mx}\, dx = \int_{-\pi}^{\pi} a_m\cdot\cos mx\cdot\cos mx\,dx = a_m\cdot\pi$$

$$\int_{-\pi}^{\pi} f(x)\cdot\sin nx\,dx = \int_{-\pi} b_n\cdot\sin nx\cdot\sin nx\,dx = b_n\cdot\pi$$

Def

Let $f(x)$ be a periodic, piecewise continuous function with period 2π.

The numbers.

$$a_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos kx \, dx, \quad k \geq 0.$$

$$b_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx, \quad k \geq 1$$

are called the Fourier coefficients of $f(x)$. The series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

is called the Fourier series of $f(x)$.

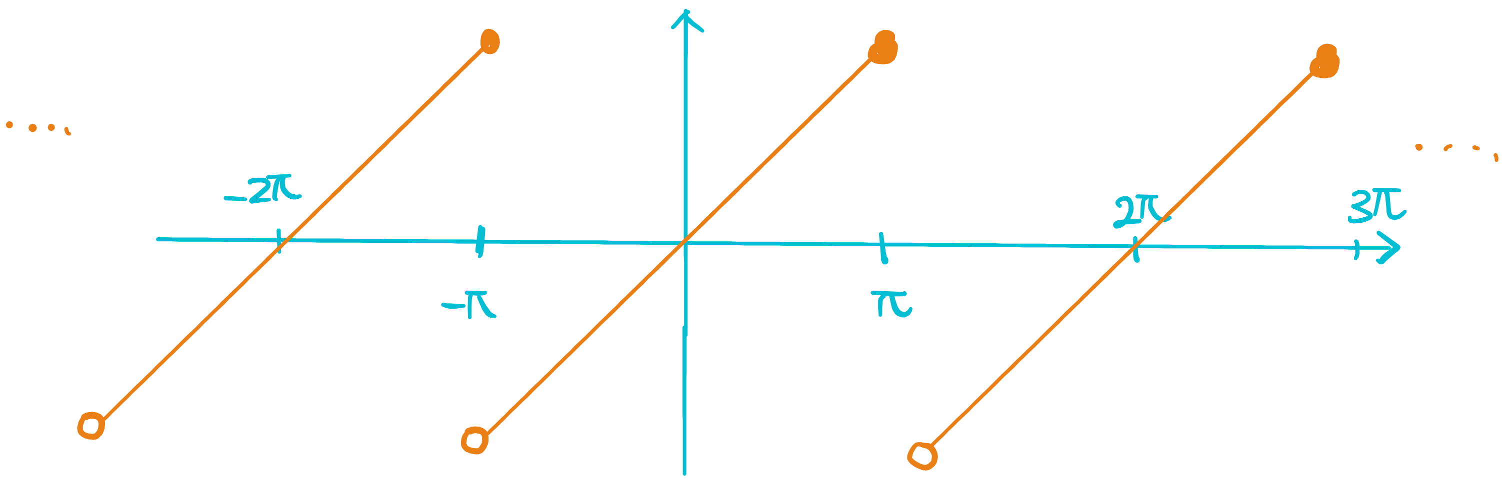
We denote this relationship by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

Example

Let $f(x)$ be the periodic function with $\overbrace{\text{period}}^{2\pi}$

S.T. $\quad f(x) = x, \qquad -\pi < x \le \pi$



odd function

odd function

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \cos kx \, dx$$

even function

$$= 0 \qquad \forall k \ge 0$$

even function

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin kx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin kx \, dx$$

Recall

$$\int_a^b u(x)\cdot v'(x)\,dx = u(x)\cdot v(x)\Big|_a^b - \int_a^b v(x)\,u'(x)\,dx$$

$v'(x) = \sin kx \qquad u'(x) = 1$

$v(x) = \frac{-\cos kx}{k} \qquad u(x) = x$

$$= \frac{2}{\pi}\left(-\frac{\cos kx}{k}\cdot x\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos kx}{k}\,dx\right)$$

$$\frac{\sin kx}{k^2}\Big|_{x=0}^{\pi} = 0$$

$$= \frac{2}{\pi} \cdot \left(- \frac{(-1)^k}{k} \cdot \pi\right) = \frac{2(-1)^{k+1}}{k} \quad \forall k > 0$$

So

$$f(x) \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx \quad \#$$

Similarly to Taylor series, we ask:

Q1: For which values of x does

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

converge?

Q2: If it converges, what is the value of sum ($\overset{?}{=} f(x)$)?

<u>Def</u>

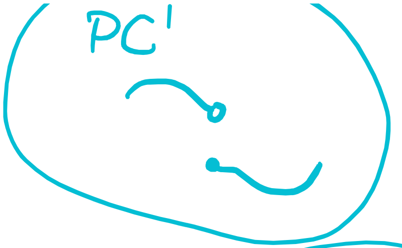
A function f is <u>piecewise continuous differentiable</u> on $\mathbb{R}$, denoted by $f \in PC^1$, if $\forall x_0 \in \mathbb{R}$, one of the following holds:

(i) f and f' are continuous at x_0

(ii) The one-sided limits

$$f(x_0^+) := \lim_{x \to x_0^+} f(x)$$

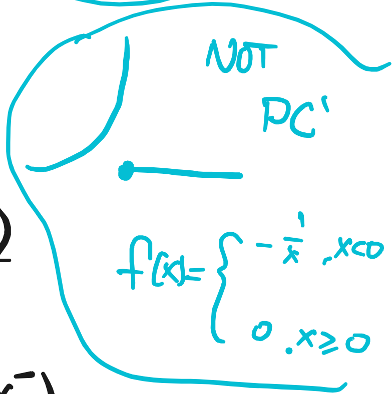
$$f(x_0^-) := \lim_{x \to x_0^-} f(x)$$



exist, and

$$f'(x_0^+) := \lim_{u \to 0^+} \frac{f(x_0+u) - f(x_0^+)}{u}$$

$$f'(x_0^-) := \lim_{u \to 0^-} \frac{f(x_0+u) - f(x_0^-)}{u}$$



exist

<u>Thm</u> (Pointwise Convergence Thm)

Let $f(x)$ be a periodic function with period 2π, and $f \in PC^1$. If

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

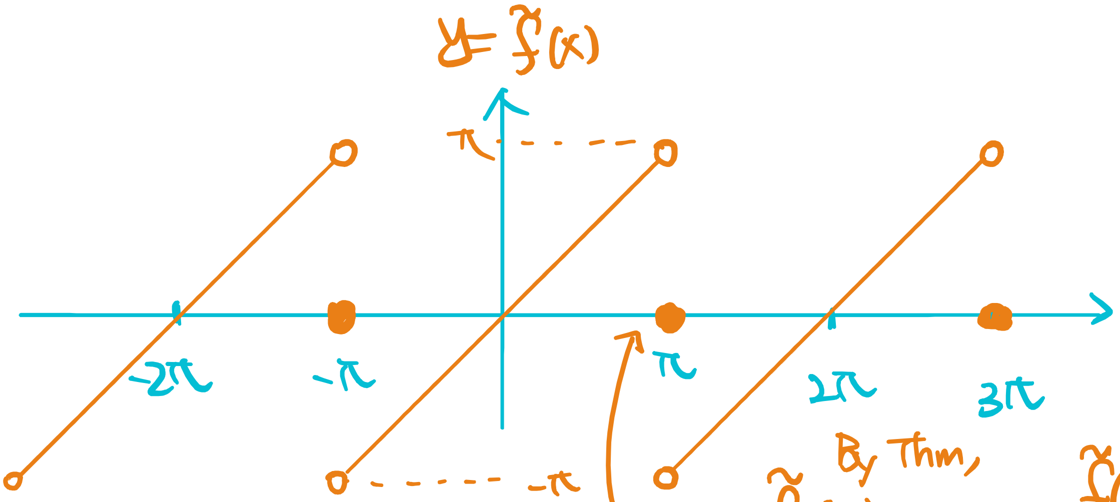
then the series converges, and

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \frac{f(x_0^+) + f(x_0^-)}{2}$$

<u>Example</u>

Recall if $f(x)$ is the 2π-periodic function s.t. $f(x) = x \quad \forall x \in (-\pi, \pi]$, then

$$f(x) \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx =: \tilde{f}(x)$$



By Thm,

$$\tilde{f}(\pi) = 0 = \frac{\tilde{f}(\pi^-) + \tilde{f}(\pi^+)}{2}$$

$$= \frac{\pi + (-\pi)}{2}$$

To prove Pointwise Conv. Thm, we need:

<u>Lemma</u> (Riemann-Lebesgue)

If f is a piecewise continuous function on $[a, b]$, then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \cdot \sin \lambda x \, dx = 0$$

$$\lim_{\lambda \to \infty} \int_a^b f(x) \cdot \cos \lambda x \, dx = 0$$

where $\lambda \in \mathbb{R}$.

<u>pf of Pointwise Conv. Thm.</u>

Let

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

Recall that

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos kx \, dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

$$\Rightarrow S_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{k=1}^{n}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt \, dt \cdot \cos kx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt \, dt \cdot \sin kx\right)$$

$$= \frac{1}{\pi}\cdot\left(\frac{1}{2}\int_{-\pi}^{\pi} f(t)\,dt + \int_{-\pi}^{\pi} f(t)\cdot\Big(\sum_{k=1}^{n} \overset{\cos(kt-kx)}{\overset{\parallel}{\boxed{\cos kx\cdot\cos kt + \sin kx\cdot\sin kt}}}\Big)\,dt\right)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\overset{u+x}{\overset{\parallel}{t}})\cdot\left(\frac{1}{2}+\sum_{k=1}^{n}\cos(\overset{ku}{\overset{\parallel}{kt-kx}})\right)dt$$

Let $u = t - x \Rightarrow du = dt$

$$= \frac{1}{\pi}\int_{-\pi\cancel{-x}}^{\pi\cancel{-x}} f(u+x)\cdot\Big(\underbrace{\boxed{\frac{1}{2}+\sum_{k=1}^{n}\cos ku}}_{}\Big)\,du \quad \Leftarrow \text{ 2π-periodic}$$

$$\overset{D_n(u)}{\parallel}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u+x)\cdot D_n(u)\,du$$

where $D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} \cos ku$ is called the Dirichlet kernel.

Recall:

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\to \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$

$\alpha = ku,\ \beta = \frac{u}{2}$

Note that

$$2\sin\left(\frac{u}{2}\right)\cdot D_n(u)$$

$$= \sin\left(\frac{u}{2}\right) + \sum_{k=1}^{n} 2\sin\left(\frac{u}{2}\right)\cdot\cos(ku)$$

$$= \sin\left(\frac{u}{2}\right) + \sum_{k=1}^{n}\left(\sin\left(\left(k+\tfrac{1}{2}\right)u\right) - \sin\left(\left(k-\tfrac{1}{2}\right)u\right)\right)$$

$$= \cancel{\sin\left(\frac{u}{2}\right)} + \left(\cancel{\sin\frac{3}{2}u} - \cancel{\sin\frac{u}{2}}\right) + \left(\cancel{\sin\frac{5}{2}u} - \cancel{\sin\frac{3}{2}u}\right)$$
$$+ \left(\cancel{\sin\frac{7}{2}u} - \cancel{\sin\frac{5}{2}u}\right) + \cancel{\cdots}$$
$$+ \sin\left(\left(n+\tfrac{1}{2}\right)u\right) - \cancel{\sin\left(\left(n-\tfrac{1}{2}\right)u\right)}$$

$$= \sin\left(\left(n+\tfrac{1}{2}\right)u\right)$$

$$\Rightarrow \boxed{D_n(u) = \frac{\sin\left(\left(n+\frac{1}{2}\right)u\right)}{2\sin\frac{u}{2}}} \qquad ①$$

Also:

② $D_n(u)$ is even

$\cos(0x)$ 1

③ $$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)\,du = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos(ku)\right)du$$

$$= 1.$$

So

$$\frac{f(x_0^+)+f(x_0^-)}{2} = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x_0^+)+f(x_0^-)}{2}\cdot D_n(u)\,du$$

$$= \frac{1}{\cancel{2}\pi}\left(\underbrace{\int_{-\pi}^{\pi} f(x_0^+)\cdot \overbrace{D_n(u)}^{\text{even}}\,du}_{=\cancel{2}\int_0^{\pi} f(x_0^+)\cdot D_n(u)\,du} + \underbrace{\int_{-\pi}^{\pi} f(x_0^-)\cdot D_n(u)\,du}_{=\cancel{2}\int_0^{\pi}\cdots}\right)$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x_0^+)D_n(u)\,du + \frac{1}{\pi}\int_0^{\pi} f(x_0^-)D_n(u)\,du$$

and

$$S_n(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+u)\cdot D_n(u)\,du$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x_0+u)\cdot D_n(u)\,du + \frac{1}{\pi}\int_{-\pi}^{0} f(x_0+u)\cdot D_n(u)\,du$$

Let $v=-u \Rightarrow dv=-du$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(x_0-v)\cdot \underbrace{D_n(-v)}_{=D_n(v)}\cdot(-dv)$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x_0-u)\cdot D_n(u)\,du$$

$\Rightarrow$

$$S_n(x_0) - \frac{f(x_0^+)+f(x_0^-)}{2}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(f(x_0+u)+f(x_0-u)-f(x_0^+)-f(x_0^-)\right)\cdot \boxed{D_n(u)}\,du$$

$$\parallel \; ①$$

$$\frac{\sin((n+\frac{1}{2})u)}{2\sin\frac{u}{2}}$$

$$= \frac{1}{2\pi}\int_0^{\pi}\overbrace{\left(\frac{f(x_0+u)+f(x_0-u)-f(x_0^+)-f(x_0^-)}{u}\cdot\frac{2\cdot\frac{u}{2}}{\sin\frac{u}{2}}\right)}^{g(u)}\cdot\sin((n+\tfrac{1}{2})u)\,du$$

Recall: $\lim\limits_{u\to 0}\frac{\sin u}{u} = 1$

Since $f(x)\in PC'$,

$g(u)$ is a piecewise continuous on $[0,\pi]$

By Riemann-Lebesgue Lemma,

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{\pi} g(u)\cdot\sin((n+\tfrac{1}{2})u)\,du = 0$$

$$\parallel$$

$$\lim_{n\to\infty}\left(S_n(x_0)-\frac{f(x_0^+)+f(x_0^-)}{2}\right)$$

$$\Rightarrow \lim_{n\to\infty}S_n(x_0) = \frac{f(x_0^+)+f(x_0^-)}{2} \qquad \#$$

Conclusion to the fundamental questions:

A1: The Fourier series of a PC' 2π-periodic function converges for any $x\in\mathbb{R}$.

A2: If $f \in PC'$, then its Fourier series converges to $\frac{f(x_0^+)+f(x_0^-)}{2}$ at x_0.

In particular, if f is continuous at x_0, $f \in PC'$,

then sum of its Fourier series $= f(x_0)$.

	Taylor series	Fourier series
basis functions	x^n	$\sin nx$, $\cos mx$
requirement	$f \in C^\infty$	f can be discontinuous (periodic)
Convergence	within a radius r $\overset{=}{\leftarrow}$ radius of conv	everywhere (if $f \in PC'$)
Condition for sum $= f(x)$	$\lim_{n\to\infty} R_n(x) = 0$	f is (2π-periodic, PC') continuous at x.