Calculus, Spring 2026, week 4

Recall

For a function $f(x)$, we consider

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \text{(Taylor expansion of } f(x) \text{ at } 0\text{)}$$

provided $\exists f^{(k)}(0)$ $\forall k \geq 0$ $(f^{(0)}(0) = f(0))$

Q1: For what x, does $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ converge?

Q2. $\displaystyle f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$??

Last week:

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + R_n(x),$$

where

$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some $\overset{c(x)}{\text{"}c\text{"}} \in J_x$, $\quad J_x = \begin{cases} [0, x] & \text{if } x \geq 0 \\ [x, 0] & \text{if } x < 0 \end{cases}$

A2: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \iff \lim_{n\to\infty} R_n(x) = 0$

<u>Example</u>

Step 0: $\frac{d^k}{dx^k} e^x \Big|_{x=0} = e^x\Big|_{x=0} = 1$

① $$\boxed{\begin{aligned} e^x = \exp(x) &= \sum_{k=0}^{\infty} \frac{1}{k!} x^k \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \forall x \in \mathbb{R} \end{aligned}}$$

Pf

It suffices to show that

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \forall x \in \mathbb{R}$$

Recall: $(e^x)' = e^x$

Since

$$R_n(x) = \boxed{\frac{d^{n+1}}{dx^{n+1}}(e^x)\Big|_{x=c}} \cdot \frac{1}{(n+1)!} \cdot x^{n+1}$$

$= e^x\Big|_{x=c} = e^c$

for some $c \in J_x$, $\Rightarrow c \le |c| \le |x|$

$$0 \le |R_n(x)| = \frac{e^c}{(n+1)!} \cdot |x|^{n+1}$$

$$\le e^{|x|} \cdot \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \text{ as } n \to \infty$$

$\forall x \in \mathbb{R}$

Recall $\lim_{n\to\infty} \frac{a^n}{n!} = 0$

(Since $\sum_{n=0}^{\infty} \frac{a^n}{n!}$ converges by Ratio Test)

$$\frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} = \frac{a}{n+1} \to 0 < 1 \ \#$$

$$\Rightarrow \lim_{n\to\infty} |R_n(x)| = 0$$

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$$

② $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \forall x \in \mathbb{R}$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$0 + \frac{1}{1!}x + \frac{0}{2!} \cdot x^2 + \frac{(-1)}{3!} x^3 + \cdots$

Step 0: $\frac{d^k}{dx^k}(\sin x)\Big|_{x=0} = \begin{cases} 0 & k \equiv 0 \pmod 4 \\ 1 & k \equiv 1 \pmod 4 \\ 0 & k \equiv 2 \pmod 4 \\ -1 & k \equiv 3 \pmod 4 \end{cases}$

$(\sin x)' = \cos x, \quad (\sin x)'' = (\cos x)' = -\sin x$

$(\sin x)''' = -\cos x \quad (\sin x)'''' = \sin x, \cdots$

pf

It suffices to show $\lim_{n\to\infty} R_n(x) = 0 \quad \forall x \in \mathbb{R}$.

$$|R_n(x)| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{d^{n+1}}{dx^{n+1}}(\sin x)\Big|_{x=c} \right|$$

Note: $\left|\frac{d^{n+1}}{dx^{n+1}}(\sin x)\right| \le 1$

$$\le \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as } n\to\infty \quad \forall x \in \mathbb{R}$$

#

③ Similarly, we have

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad \forall x \in \mathbb{R}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Taylor expansions at a: $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Thm (Thm 12.7.1)

If f has $n+1$ continuous derivatives on $I = (a-\delta, a+\delta)$, then $\forall x \in I$,

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)\cdot(x-t)^n\,dt$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some $c = c(x)$ between a and x.

Example

The polynomial

$$f(x) = 4x^3 - 3x^2 + 5x - 1$$

can be expanded in powers of $(x-2)$:

$$f^{(0)}(2) = f(2) = 29$$

$$f'(2) = 12x^2 - 6x + 5\Big|_{x=2} = 41$$

$$f''(2) = 24x - 6\Big|_{x=2} = 42$$

$$f'''(2) = 24, \quad \left(f^{(n)}(2) = 0 \ \forall n \geq 4\right)$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k \qquad \left(\Rightarrow \lim_{n\to\infty} R_n(x) = 0\right)$$

$$= 29 + 41(x-2) + \frac{42}{2!}(x-2)^2 + \frac{24}{3!}(x-2)^3$$

$$= f(x)$$

So $4x^3 - 3x^2 + 5x - 1$

$$= 29 + 41(x-2) + 21(x-2)^2 + 4(x-2)^3$$

#

§ Power series

We consider

$$\sum_{k=0}^{\infty} a_k x^k \qquad \left(\text{or } \sum_{k=0}^{\infty} a_k (x-a)^k\right)$$

without a given $f(x)$.

For simplicity, we discuss $\sum_{k=0}^{\infty} a_k x^k$

All the parallel definitions and results hold for $\sum_{k=0}^{\infty} a_k (x-a)^k$

Def

A power series $\sum_{k=0}^{\infty} a_k x^k$

(i) converges at c if $\sum_{k=0}^{\infty} a_k c^k$

converges

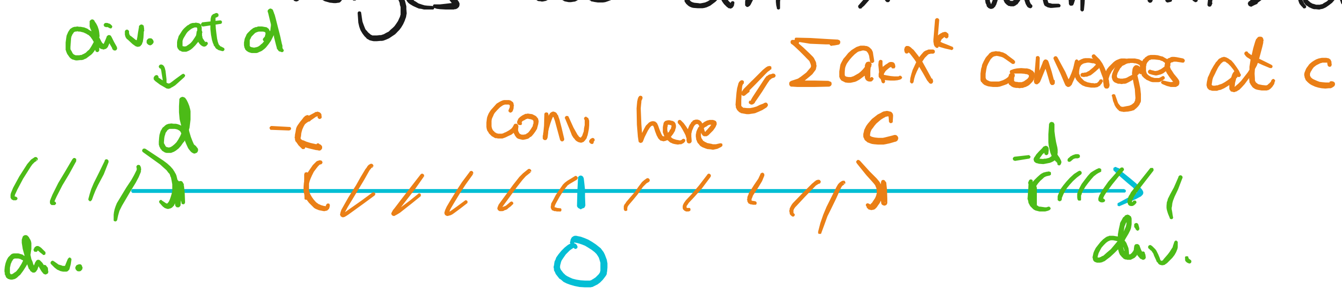
(ii) Converges on a set S if

$$\sum_{k=0}^{\infty} a_k c^k \text{ Converges } \quad \forall c \in S.$$

Thm (Thm 12.8.2)

(i) If $\sum_{k=0}^{\infty} a_k x^k$ Converges at $c \neq 0$, then it converge absolutely at all x with $|x| < |c|$.

(ii) If $\sum_{k=0}^{\infty} a_k x^k$ diverges at d, then it diverges at all x with $|x| > |d|$



pf

It suffices to show (i):

Suppose $\sum_{k=0}^{\infty} a_k c^k$ Converges $\overset{n^{th} \text{ Term Test}}{\Longrightarrow} \lim_{k\to\infty} a_k c^k = 0$

$\Rightarrow$ For $\varepsilon = 1 > 0$, $\exists$ $N = N_1$ s.t.

$$|a_k c^k| = |a_k c^k - 0| < 1 \qquad \forall k \geq N_1$$

$$\Rightarrow \quad |a_k x^k| = |a_k c^k| \cdot \left|\frac{x}{c}\right|^k$$

$$< \left|\frac{x}{c}\right|^k \qquad \forall\, k \geq N_1$$

If $|x| < |c|$, then $\left|\frac{x}{c}\right| < 1$

$$\Rightarrow \quad \sum_{k=0}^{\infty} \left|\frac{x}{c}\right|^k \quad \text{Converges}$$

By the basic comparison test,

$$\sum_{k=0}^{\infty} |a_k x^k| \quad \text{Converges}$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k x^k \quad \text{Converges absolutely}$$

#

Let $r = \sup \left\{ x \in \mathbb{R} \;\middle|\; \sum_{k=0}^{\infty} a_k x^k \text{ converges} \right\}$

By the previous thm, $\exists$ 3 cases:

<u>Case 1 ($r=0$)</u>:

$\sum_{k=0}^{\infty} a_k x^k$ Converges only at $x=0$

e.g. $\sum_{k=1}^{\infty} k^k \cdot x^k$ <u>Note</u>: $\sqrt[k]{|k^k x^k|} = k \cdot |x| \xrightarrow{\text{if } x \neq 0} \infty$ as $k \to \infty$

Root Test $\Rightarrow$ it diverges.

case 2 ($r=\infty$):

$\sum_{k=0}^{\infty} a_k x^k$ converges absolutely at each $x \in \mathbb{R}$.

e.g. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ $(= e^x \quad \forall x \in \mathbb{R})$

case 3 ($r \neq 0, \infty$):

$\exists\, r > 0$ s.t.

① $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for $|x| < r$

② $\sum_{k=0}^{\infty} a_k x^k$ diverges for $|x| > r$.

e.g. $\sum_{k=0}^{\infty} x^k$ $\begin{cases} \text{converges absolutely for } |x|<1 \\ \text{diverges for } |x| > 1 \end{cases}$

Def

The radius of convergence (收斂半徑) of $\sum_{k=0}^{\infty} a_k x^k$ is

(i) 0 if it is case 1

(ii) ∞ if it is case 2

(iii) r if it is case 3 ($\Rightarrow$ it converges only $(-r, r)$)

The interval of convergence of $\sum_{k=0}^{\infty} a_k x^k$

is the maximal interval on which it converges

<u>Example</u>

① $\sum_{k=1}^{\infty} k^k x^k$: radius of convergence $= 0$
interval 〃 $= \{0\}$

② $\sum_{k=0}^{\infty} \frac{x^k}{k!}$: radius of convergence $= \infty$
interval 〃 $= (-\infty, \infty)$

③ $\sum_{k=0}^{\infty} x^k$: radius of conv. $= 1$
interval 〃 $= (-1, 1)$

<u>Note</u>: $\sum_{k=0}^{\infty} 1^k$ and $\sum_{k=0}^{\infty} (-1)^k$ diverge

By Root Test, we have:

<u>Thm</u>

The radius of convergence of $\sum_{k=0}^{\infty} a_k x^k$

$$= \frac{1}{\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}}$$

pf

By Root Test,

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} \cdot |x| < 1$$

$$\Leftrightarrow \quad |x| < \frac{1}{\overline{\lim} \sqrt[k]{|a_k|}}$$

$\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k x^k|} \begin{cases} < 1 \Rightarrow \sum |a_k x^k| \text{ conv.} \\ > 1 \Rightarrow \sum |a_k x^k| \text{ div.} \end{cases}$ $\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}$

$\Leftrightarrow |x| > \dfrac{1}{\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}}$ .....

Note that if $|x| > \dfrac{1}{\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}}$, then

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k x^k|} > 1$$

Recall if $\overline{\lim_{n\to\infty}} b_n = \beta$ then for $\varepsilon > 0$, $\exists (b_{n_j})$ s.t. $b_{n_j} > \beta - \varepsilon$ $\forall j \geq 1$

$$\Rightarrow \exists \sqrt[k_j]{|a_{k_j} x^{k_j}|} > 1 \quad \forall j \geq 1$$

take $\varepsilon = \left(\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k x^k|} - 1\right)/2 > 0$

$$\Rightarrow \lim_{k\to\infty} |a_k x^k| \neq 0$$

(Otherwise, for $\varepsilon = \frac{1}{2} > 0$, $\exists N_\varepsilon$ s.t. $|a_k x^k| < \frac{1}{2}$ $\forall k \geq N_\varepsilon$ $\Rightarrow \sqrt[k]{|a_k x^k|} < \sqrt[k]{\frac{1}{2}} < 1$

$$\Rightarrow \sum_{k=0}^{\infty} a_k x^k \text{ diverges}$$

#

$\underline{\lim} \left|\frac{a_{k+1}}{a_k}\right| \leq \underline{\lim} \sqrt[k]{|a_k|} \leq \overline{\lim} \sqrt[k]{|a_k|} \leq \overline{\lim} \left|\frac{a_{k+1}}{a_k}\right|$

Recall:

Remark

If the limit

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

(This implies that $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L$)

exists, then the radius of conv. of

$\sum_{k=0}^{\infty} a_k x^k$ is $\frac{1}{L}$.

Example

Find the interval of conv. of the following power series:

① $\sum_{k=1}^{\infty} \frac{x^k}{k}$:

Step1: Compute the radius of conv.

$$r = \frac{1}{\overline{\lim_{k\to\infty}} \sqrt[k]{\left|\frac{1}{k}\right|}} = \overline{\lim_{k\to\infty}} \sqrt[k]{k} = 1$$

$$\| \quad \frac{1}{1} \quad \Uparrow$$

or by Ratio Test,

$$\lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k/k}{(k+1)/k} = 1$$

$\left(\Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k}\right.$ conv. for $x \in (-1, 1)$

and div. for $|x| > 1 \Big)$

Step2: Check the convergence at endpoints:

$x = 1$: $\sum_{k=1}^{\infty} \frac{x^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ div. (p-series)

$x=-1$: $\sum_{k=1}^{\infty} \frac{x^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ conv.

(Alternating Series Test)

<u>Conclusion</u>

The interval of conv. of $\sum_{k=1}^{\infty} \frac{x^k}{k}$ is $[-1, 1)$. #

② $\sum_{k=1}^{\infty} \frac{k}{6^k} x^k$:

<u>Step1</u>:

$$r = \frac{1}{\overline{\lim_{k\to\infty}} \sqrt[k]{\frac{k}{6^k}}} = \frac{1}{\lim_{k\to\infty} \frac{\sqrt[k]{k}}{6}} = 6$$

<u>Step2</u>:

$x=6$: $\sum_{k=1}^{\infty} \frac{k}{6^k} x^k = \sum_{k=1}^{\infty} \frac{k}{\cancel{6^k}} \cancel{6^k} = \sum_{k=1}^{\infty} k$ div.

$x=-6$: $\sum_{k=1}^{\infty} \frac{k}{6^k} x^k = \sum_{k=1}^{\infty} (-1)^k \cdot k$ div. since $\lim_{k\to\infty} (-1)^k \cdot k \neq 0$

$\Rightarrow$ interval of conv. $= (-6, 6)$ #

③ $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k} (x+2)^k$

Step1:

$$r = \frac{1}{\overline{\lim_{k\to\infty}} \sqrt[k]{\left| \frac{(-1)^k}{k^2 3^k} \right|}} = \frac{3}{\overline{\lim_{k\to\infty}} \frac{1}{(\sqrt[k]{k})^2}} = 3$$

Step2:

$x = -2+3 = 1$:

(alternating series)

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k}(x+2)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k} \cdot 3^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \text{ Conv.}$$

$x = -2-3 = -5$:

(p-series)

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k}(x+2)^k = \sum_{k=1}^{\infty} \frac{\cancel{(-1)^k}}{k^2 \cancel{3^k}} \cancel{(-3)^k} = \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ Conv.}$$

Conclusion

interval of Conv. $= [-5, 1]$ #

Thm (Thm 12.9.1 – 12.9.3)

Suppose r is the radius of Conv. of $\sum_{k=0}^{\infty} a_k x^k$.

(i) Radius of conv. of $\sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$ is also r.

(ii) Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Then

$f(x) \in C^{\infty}(-r, r)$ 可微 ∞ 次

and

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \quad \forall x \in (-r, r)$$

(iii) If $[a,b] \subseteq (-r, r)$, then

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty} \int_a^b a_k x^k \,dx$$

<u>Example</u>

Recall

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

on $(-1, 1)$.

$$\Rightarrow^{①} \left(\frac{1}{1-x}\right)' = \frac{-(1-x)'}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$= \sum_{k=1}^{\infty} k \cdot x^{k-1} = 1 + 2x + 3x^2 + \cdots$$

So $$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k \cdot x^{k-1} \quad \forall x \in (-1,1)$$

$$\left((1-x)^{-2}\right)' = -2\cdot(1-x)^{-3}\cdot(-1)$$

② $$\left(\frac{1}{(1-x)^2}\right)' = \frac{2}{(1-x)^3}$$

$$= \sum_{k=2}^{\infty} k\cdot(k-1)\,x^{k-2} = 2+6x+12x^2+\cdots$$

So $$\frac{1}{(1-x)^3} = \sum_{k=2}^{\infty} \frac{k(k-1)}{2}\,x^{k-2} \quad \forall x\in(-1,1)$$

Example

① $$\sum_{k=1}^{\infty} \frac{x^k}{k} = ? \qquad (x\in(-1,1))$$

Sol

$$\sum_{k=0}^{\infty} t^k = \frac{1}{1-t} \qquad \forall t\in(-1,1)$$

$\Rightarrow$ For $x\in(-1,1)$, $[0,x]$ or $[x,0] \subseteq (-1,1)$

$$\int_0^x \sum_{k=0}^{\infty} t^k\,dt \overset{\text{Thm}}{=} \sum_{k=0}^{\infty} \int_0^x t^k\,dt$$

$$\left.\frac{t^{k+1}}{k+1}\right|_0^x = \frac{x^{k+1}}{k+1}$$

$$= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\left(-\ln(1-t)\right)' = \frac{1}{1-t}$$

$$= \int_0^x \frac{1}{1-t}\,dt = -\ln(1-t)\Big|_{t=0}^{x}$$

$$= -\ln(1-x) + \cancel{\ln 1}^{=0} = \ln\frac{1}{1-x}$$

So $\ln\frac{1}{1-x} = \sum_{k=1}^{\infty}\frac{x^k}{k} \quad \forall x\in(-1,1)$ #

② $\boxed{\ln(1+x) \overset{?}{=} \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1}}$ Taylor expansion of ln

<u>Sol</u>

Note: $(\ln(1+x))' = \frac{1}{1+x} = \sum_{k=0}^{\infty}(-x)^k \quad \forall x\in(-1,1)$

So $\ln(1+x) - \underbrace{\cancel{\ln(1+0)}}_{=0} = \int_0^x (\ln(1+t))'\,dt$

$$= \int_0^x \sum_{k=0}^{\infty}(-t)^k\,dt \qquad \left(\frac{(-1)^k t^{k+1}}{k+1}\right)'$$

for $x\in(-1,1)$

$$= \sum_{k=0}^{\infty}\int_0^x (-t)^k\,dt$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\cdot x^{k+1}$$ #

Q: What if $x=\pm 1$?

<u>Note</u>: $\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1}$ conv. on $(-1,1]$

Q: $\ln(1+1)=\ln 2 \overset{?}{=} \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$

<u>Thm</u> (Abel's Thm, Thm 12.9.5)

Suppose that $\sum_{k=0}^{\infty} a_k x^k$ Converges on $(-c, c)$ and $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $(-c, c)$.

(i) If $\lim_{x \to c^-} f(x)$ exists, $= f(c)$, then

$$f(c) = \sum_{k=0}^{\infty} a_k c^k$$

(ii) If $\lim_{x \to -c^+} f(x)$ exists, $= f(-c)$, then

$$f(-c) = \sum_{k=0}^{\infty} a_k (-c)^k$$

<u>Cor</u>

$f(x) = \ln(1+x)$

$\lim_{x \to 1^-} f(x) \overset{\text{exists}}{=} \ln(1+1) = \ln 2$

For $x \in (-1, 1]$,

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$= \ln 2$$

<u>Cor</u>

For $x \in [-1, 1]$,

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

pf

Recall $(\tan^{-1}x)' = \dfrac{1}{1+x^2} = \sum_{k=0}^{\infty}(-x^2)^k$

$\forall x \in (-1,1)$

$$\Rightarrow \tan^{-1}x - \underbrace{\cancel{\tan^{-1}0}}_{=0} = \int_0^x (\tan^{-1}t)'\,dt$$

for $x\in(-1,1)$,
$$= \int_0^x \sum_{k=0}^{\infty}(-t^2)^k\,dt$$

$$(-1)^k t^{2k} = \left((-1)^k \frac{t^{2k+1}}{2k+1}\right)'$$

$$= \sum_{k=0}^{\infty}\int_0^x \underbrace{(-t^2)^k}_{=}\,dt$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}$$

So
$$\tan^{-1}x = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1} \quad \forall x\in(-1,1)$$

Since $\tan^{-1}x$ is continuous $(-\frac{\pi}{2},\frac{\pi}{2}) \supseteq [-1,1]$

" " at $x=\pm1$

Thm $\Rightarrow$
$$\tan^{-1}x = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1} \quad \forall x\in[-1,1]$$

#

Example

$f(x)$ //

$$\left.\frac{d^{10}\left(x^2\cos(x^3)\right)}{dx^{10}}\right|_{x=0} = ?$$

Sol

x^{6k}

$$x^2 \underline{\cos(x^3)} = x^{2} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \underbrace{(x^3)^{2k}}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad \forall x \in (-\infty, \infty)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{6k+2} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$=10 \Rightarrow k = \frac{8}{6} \notin \mathbb{Z}$

$$\Rightarrow \quad f^{(10)}(0) = 0$$

#

1. Determine whether the series converges or diverges. Explain why.

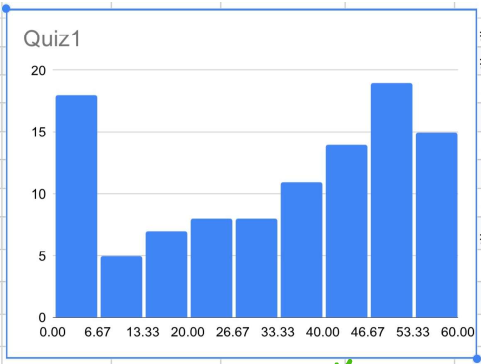
   a. $\sum_{k=2}^{\infty} \frac{k}{k^3-k}$.

   b. $\sum_{k=1}^{\infty} \frac{1}{1+2\ln k} \geq \frac{1}{1+2k}$ Comparison test

   c. $\sum_{k=1}^{\infty} \frac{k^k}{3^{k^2}}$.

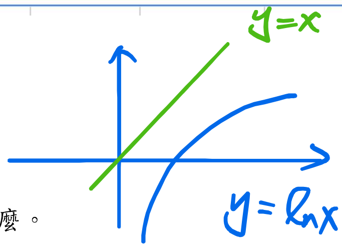
   d. $\sum_{k=1}^{\infty} \frac{2\cdot 4\cdots 2k}{(2k)!}$.

Average	35.67226891
Average (except zero)	41.61764706
Quartile 0	0
Quartile 1	19
Quartile 2	40
Quartile 3	53
Quartile 4	60
標準差	20.42020819
非零數	102
不到50%人數	39
滿分人數	14



$\frac{1}{\ln k} \geq \frac{1}{k}$, $\sum \frac{1}{k}$ diverges

易錯：



- $\lim_{k\to\infty} a_k = 0$ 無法推得無窮級數 $\sum a_k$ 收斂。
- (b) 小題須注意 $\sum \frac{1}{\ln k}$ 是發散級數。
- (c) 跟 (d) 小題使用 ratio test 必須計算極限 $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$。單就 $\frac{a_{k+1}}{a_k} < 1$ 無法得出什麼。

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2. In this problem, we consider the following question: Suppose we know that a sequence $(a_n)_{n=1}^{\infty}$ satisfies the condition that for any $\epsilon > 0$, there exists $N = N_\epsilon \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$ for all $n \geq N$. Can we conclude that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence?

   $\sum_{k=1}^{\infty} \frac{1}{k^p}$ $p = \frac{1}{3} < 1$

   a. Let $a_n = \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}}$. Prove that for any $\epsilon > 0$, there exists $N = N_\epsilon \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \epsilon$ for all $n \geq N$.

   b. Prove that the above sequence $(a_n)_{n=1}^{\infty}$ is not a Cauchy sequence.

易錯：

- (b) 小題中 Cauchy sequence 的反敘述是

$$\exists\, \epsilon > 0 \text{ such that } \forall\, N > 0,\ \exists\, m, n \in \mathbb{N} \text{ such that } m, n > N \text{ and } |a_m - a_n| \geq \epsilon.$$

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3. Suppose that the series $\sum_{k=1}^{\infty} a_k^2$ is convergent, where $a_k \geq 0$.

   a. Is $\sum_{k=1}^{\infty} a_k$ necessarily convergent?

   $$\left|\sum_{n=1}^{N} a_n b_n\right| \leq \left(\sum_{n=1}^{N} a_n^2\right)\cdot\left(\sum_{n=1}^{N} b_n^2\right)$$

   b. Prove that $\sum_{k=1}^{\infty} \frac{a_k}{k}$ is convergent.

注意：

$$\sum_{k=1}^{n} \frac{a_k}{k} \leq \left(\sum_{k=1}^{n} a_k^2\right)\cdot\left(\sum_{k=1}^{n}\left(\frac{1}{k}\right)^2\right) < M \text{ for some } M$$

- (b) 小題直接採用 ℓ^2 中的 Cauchy 不等式是有問題的。原因是只有 $\sum_{k=1}^{\infty} \frac{a_k}{k}$ 收斂後才能執行比大小。

increasing (a) Conv.

bounded aboveе

$\Rightarrow$ conv.