Calculus, Spring 2026, week 3

Recall

$\sum a_k$ is absolutely convergent if
絕對 收斂
$\sum |a_k|$ is convergent

$\sum a_k$ is conditionally convergent if
條件 收斂

$\sum a_k$ is convergent, but not absolutely convergent

Thm

$\sum |a_k|$ converges $\Rightarrow$ $\sum a_k$ converges.

Pf

Assume $\sum_{k=1}^{\infty} |a_k|$ converges

$\Rightarrow \sum_{k=1}^{\infty} 2|a_k|$ also converges

Note that

$$0 \le a_k + |a_k| \le |a_k| + |a_k| = 2|a_k|$$

By the basic comparison test,

$\sum_{k=1}^{\infty} (a_k + |a_k|)$ converges.

$$\Rightarrow \sum_{k=1}^{\infty}(a_k+|a_k|) - \sum_{k=1}^{\infty}|a_k|$$

$$= \sum_{k=1}^{\infty} a_k \quad \text{Converges.} \qquad \#$$

Example

① $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1-\frac{1}{4}+\frac{1}{9}-\cdots$ is (absolutely) convergent

because $\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{k^2}\right|$ converges.

② $\frac{1}{2}-\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}-\frac{1}{2^5}-\frac{1}{2^6}+\cdots$

is (absolutely) convergent

③ $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = \frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

is NOT absolutely convergent because

$$\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{k}\right| = \sum_{k=1}^{\infty}\frac{1}{k} \quad \text{diverges.}$$

Q: Is $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ convergent?

We need the follow lemma for establishing criteria for convergence of $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$.

Lemma

Let $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ be seq. s.t.

(i) $(b_k)_{k=1}^{\infty}$ is monotone,

(ii) $\exists\, M>0$ s.t. $|a_1+a_2+\cdots+a_k| \le M \quad \forall k\in\mathbb{N}$

Then $\forall n\in\mathbb{N}$,

$$|a_1b_1+a_2b_2+\cdots+a_nb_n| \le M(|b_1|+2|b_n|)$$

pf

$(\Rightarrow a_n = S_n - S_{n-1})$

Let $\underbrace{S_0=0}$, $S_n = a_1+\cdots+a_n$

$$a_1b_1+\cdots+a_nb_n = (S_1-S_0)b_1+(S_2-S_1)b_2 + \cdots + (S_n-S_{n-1})b_n$$

$$= \underbrace{\cancel{(-b_1)S_0}}_{=0} + \underline{(b_1-b_2)}S_1 + \underline{(b_2-b_3)}S_2 + \cdots + \underline{(b_{n-1}-b_n)}S_{n-1} + b_nS_n$$

Since (b_k) is monotone,

$$(b_1-b_2), (b_2-b_3), \cdots, (b_{n-1}-b_n)$$

are either all nonnegative or all non-positive

$\Rightarrow$

$$\underline{|(b_1-b_2)|\cdot|S_1| + \cdots + |(b_{n-1}-b_n)|\cdot|S_{n-1}|}$$

$$= \left| (b_1-b_2)\cdot|S_1| + \cdots + (b_{n-1}-b_n)\cdot|S_{n-1}| \right|$$

by (ii) $\leq \left| (b_1 - \cancel{b_2}) \cdot M + \overset{(\cancel{b_2} - \cancel{b_3})M}{\cancel{\cdots}} + (\cancel{b_{n-1}} - b_n) \cdot M \right|$

$$= \underline{|b_1 - b_n| \cdot M}$$

So $|a_1 b_1 + \cdots + a_n b_n| = \left| (b_1 - b_2) S_1 + \cdots + (b_{n-1} - b_n) S_{n-1} + b_n S_n \right|$

$$\leq \underline{|b_1 - b_2| \cdot |S_1| + |b_2 - b_3| \cdot |S_2| + \cdots + |b_{n-1} - b_n| \cdot |S_{n-1}|} + |b_n| \cdot \underbrace{|S_n|}_{\leq M \ \text{(ii)}}$$

$$\leq |b_1 - b_n| \cdot M + |b_n| \cdot M$$

$$\leq (|b_1| + |b_n|) \cdot M + |b_n| \cdot M$$

$$= (|b_1| + 2|b_n|) \cdot M \qquad \#$$

<u>Thm</u> (Dirichlet's Test)

Let (a_k), (b_k) be seq. Suppose

(i) $\exists M > 0$ s.t. $\left| \sum_{k=1}^{n} a_k \right| \leq M \quad \forall n \in \mathbb{N}$,

(ii) (b_k) is monotone,

(iii) $\lim_{k \to \infty} b_k = 0$.

Then $\sum_{k=1}^{\infty} a_k b_k$ converges.

$\sum \frac{(-1)^{k+1}}{k}$

$b_k = \frac{1}{k}$, $a_k = (-1)^{k+1}$

<u>pf</u>

Let $S_n = \sum_{k=1}^{n} a_k$, $S_0 = 0$. By (i),

$$|a_{m+1} + a_{m+2} + \cdots + a_n| = |S_n - S_m|$$
$$\leq |S_n| + |S_m| \leq 2M$$

$\forall\, n, m \in \mathbb{N}$, $n > m$.

By Lemma,

$$|a_{m+1} b_{m+1} + \cdots + a_n b_n|$$
$$\leq 2M \cdot (|b_{m+1}| + 2|b_n|)$$

Since $\lim_{k\to\infty} b_k = 0$, $\forall \varepsilon > 0\ \exists N = N_\varepsilon$ s.t.

$$|b_k| = |b_k - 0| < \frac{\varepsilon}{6M} \quad \forall\, k \geq N$$

So for $n > m \geq N$,

$$|a_{m+1} b_{m+1} + \cdots + a_n b_n| \leq 2M \cdot (|b_{m+1}| + 2|b_n|)$$
$$< 2M \cdot \left(\frac{\varepsilon}{6M} + 2 \cdot \frac{\varepsilon}{6M}\right) = \varepsilon$$

This shows that $\left(\sum_{k=1}^{n} a_k b_k\right)_{n=1}^{\infty}$ is a Cauchy seq.

Recall: $\forall \varepsilon > 0\ \exists N = N_\varepsilon$ s.t. $\forall n, m \geq N$, $n > m$

$$\left|\sum_{k=1}^{n} a_k b_k - \sum_{k=1}^{m} a_k b_k\right| < \varepsilon$$

$$(= |a_{m+1} b_{m+1} + \cdots + a_n b_n|)$$

$\Rightarrow \sum_{k=1}^{\infty} a_k b_k$ Converges. #

<u>Cor</u> (Alternating Series Test, a.k.a. Leibniz's Test, Thm 12.5.3)

Let $(b_k)_{k=1}^{\infty}$ be a decreasing seq of non-negative numbers. Then

$$\sum_{k=1}^{\infty} (-1)^{k+1} \cdot b_k = b_1 - b_2 + b_3 - b_4 + \cdots$$

Converges $\Longleftrightarrow \lim_{k\to\infty} b_k = 0$

(e.g. $b_k = \frac{1}{k}$, we know $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ Converges)

Pf

"$\Rightarrow$" is from n^{th}-Term Test

"$\Leftarrow$" Let $a_k = (-1)^{k+1}$

$$\Rightarrow \sum_{k=1}^{n} a_k = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\Rightarrow \left|\sum_{k=1}^{n} a_k\right| \leq 1 \quad \forall n \in \mathbb{N} \quad (M=1)$$

By Dirichlet's Test, if $\lim_{k\to\infty} b_k = 0$, then $\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} (-1)^{k+1} b_k$ Converges #

Remark

A rearrangement of $\sum_{k=1}^{\infty} a_k$ is a series that has exactly same terms but in a different order.

For example,

$$"a_1 + a_3 + a_2" + "a_4 + a_6 + a_5" + \cdots$$

is a rearrgement $\sum_{k=1}^{\infty} a_k$.

It is a theorem of Riemann that all the rearrangements of an absolutely convergent series converge absolutely to the same limit.

In contrast, a conditionally convergent series can be rearraged to converge to any real number, to diverge to $+\infty$, to diverge to $-\infty$ or even oscillate between 2 numbers.

The following theorem is useful for the study of power series.

Thm (Abel's Test)

Suppose

(i) $\sum_{k=1}^{\infty} a_k$ Converges,

(ii) $(b_k)_{k=1}^{\infty}$ is monotone and bdd.

Then $\sum_{k=1}^{\infty} a_k b_k$ Converges.

pf

Step1 (b_k) is bdd $\Leftrightarrow \exists M>0$ s.t. $|b_k|<M \; \forall k\in\mathbb{N}$.

Step2 Since $\sum_{k=1}^{\infty} a_k$ converges, the seq.

$$\left(S_n = \sum_{k=1}^{n} a_k\right)_{n=1}^{\infty}$$

is Cauchy. $\Rightarrow \forall \varepsilon>0, \exists N=N_\varepsilon$ s.t.

$$|S_n - S_m| = |a_{m+1}+\cdots+a_n| < \frac{\varepsilon}{3M}$$

when $n>m\geq N$.

"M in Lemma"

Step3.

By Lemma,

$$|a_{m+1}b_{m+1}+\cdots+a_n b_n| \leq \frac{\varepsilon}{3M}\cdot\left(|b_{m+1}|+2|b_n|\right)$$

$$< \frac{\varepsilon}{3M} \cdot (M+2M) = \varepsilon$$

$\forall\ n > m \geq N$

So $\left(\sum_{k=1}^{n} a_k b_k\right)_{n=1}^{\infty}$ is Cauchy

$\Rightarrow \sum_{k=1}^{\infty} a_k b_k$ converges #

Cor

If $\sum_{k=1}^{\infty} a_k$ converges, then the power series

$$\sum_{k=1}^{\infty} a_k \cdot x^k$$

is convergent for all $x \in [0,1]$.

Choose $b_k = x^k$ for $x \in [0,1)$.

§ Series of functions

We have understood differentiation and integration of polynomials.

Now we try to approach more complicated

functions by polynomials:

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Let us assume they have the same derivatives at 0 up to n-th order:

$$f(0) = a_0, \qquad f'(0) = a_1$$

$$f''(0) = 2a_2 + 3\cdot 2\, a_3 x^1 + 4\cdot 3\cdot a_4 x^2 + \cdots \, n(n-1)\, a_n x^{n-2} \Big|_{x=0}$$

$$= 2a_2$$

$$\vdots$$

$$f^{(n)}(0) = n!\, a_n$$

This computation suggests to consider

$$f(0) + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n$$

"best deg-n polynomial approximation of $f(x)$ around 0"

To optimize this approach, we consider

$$f(x) \approx \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \leftarrow \text{Taylor series (Taylor expansion) at 0}$$

$$= f(0) + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \cdots$$

Questions:

(i) For which values of x, deos the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$?

(ii) If this series converges at x, is the sum equal to $f(x)$?

Thm (Taylor's Thm, Thm 12.6.1)

If $f(x)$ has $n+1$ continuous derivatives on an open interval $I = (a, b)$, $\ni 0$

then $\forall x \in I$,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

$$\circledast \qquad + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

is called the remainder.

pf (Induction on n)

$n=0$: Since

fundamental thm of Calculus

$$\int_0^x f'(t)\,dt = f(x) - f(0)$$

continuous

$$\Rightarrow \quad f(x) = f(0) + \int_0^x f'(t)\cdot 1\,dt \qquad \text{" } R_0(x)$$

$\Rightarrow$ "Case $n=0$" is ok.

Recall:

$$\int_a^b u(t)\,v'(t)\,dt = u(t)v(t)\Big|_a^b - \int_a^b v(t)\,u'(t)\,dt$$

<u>$n=1$</u>: Let $u(t) = f'(t)$, $v(t) = t - x$ $\quad$ ($v'(t) = 1$)

integration by parts

$$= f(0) + f'(t)\,(t-x)\Big|_{t=0}^{x} - \int_0^x f''(t)\cdot(t-x)\,dt$$

$$= f(0) + \cancel{f'(x)\cdot(x-x)} - f'(0)\cdot(0-x) - \int_0^x f''(t)\,(t-x)\,dt$$

$$= f(0) + f'(0)\cdot x + \boxed{\int_0^x f''(t)\cdot(x-t)\,dt} = R_1(x)$$

$\Rightarrow$ "Case $n=1$" is ok

<u>Induction step:</u>

Assume ⊛ holds for $(0, 1, 2, \dots,)\, n-1$.

That is,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \dots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

$$+ \frac{1}{(n-1)!}\int_0^x f^{(n)}(t)\cdot(x-t)^{n-1}\,dt \qquad \text{" } R_{n-1}(x)$$

Let $u(t) = f^{(n)}(t)$ $\qquad v(t) = \dfrac{-(x-t)^n}{n}$

$u'(t) = f^{(n+1)}(t)$ $\qquad v'(t) = (x-t)^{n-1}$

$$\cancel{f^{(n)}(x)\cdot\frac{-(x-x)^n}{n}} = 0$$

$$+ f^{(n)}(0)\cdot\frac{+x^n}{n}$$

$$= f(0) + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{1}{(n-1)!}\left(f^{(n)}(t)\cdot\left(\frac{-(x-t)^n}{n}\right)\Bigg|_{t=0}^{x} - \int_0^x f^{(n+1)}(t)\cdot\left(\frac{-(x-t)^n}{n}\right)dt \right)$$

$$= f(0) + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(0)}{n!}\cdot x^n + \underbrace{\boxed{\frac{1}{n!}\int_0^x f^{(n+1)}(t)\cdot(x-t)^n\,dt}}_{=R_n(x)}$$

$\Rightarrow$ $\circledast$ holds for n

$\Rightarrow$ Induction is complete $\qquad$ #

The Remainder $R_n(x)$ can be expressed in another form. To get that form, we need:

Thm 5.9.3

Thm (Second Mean Value Thm for integrals.)

Assume $u, v \in C[a,b] = \{ \text{continuous functions on } [a,b] \}$

and $v(x) \geq 0 \quad \forall x \in [a,b]$.

Then $\exists\ c \in [a,b]$ s.t.

$$\circledast \quad \int_a^b u(x) \cdot v(x)\, dx = u(c) \cdot \int_a^b v(x)\, dx$$

pf

If $V(x) = 0 \quad \forall x \in [a,b]$, then $\circledast$ holds

Assume $V \not\equiv 0 \xRightarrow{v \in C[a,b]} \quad a<b \quad \int_a^b v(x)dx > 0$

Since $u \in C[a,b]$, by the extreme value thm,

$\exists\ m \left(= \min_{x\in[a,b]} u(x) \right)$, $M = \left(\max_{x\in[a,b]} u(x) \right)$ s.t.

$m = u(x_m)$, $M = u(x_M)$

$$m \leq u(x) \leq M \qquad \forall x \in [a,b]$$

$\because v(x) \geq 0$

$$\Rightarrow \quad m \cdot v(x) \leq u(x) \cdot v(x) \leq M \cdot v(x) \quad \forall x \in [a,b]$$

$$\Rightarrow \int_a^b m \cdot v(x) dx \leq \int_a^b u(x) v(x) dx \leq \int_a^b M \cdot v(x) dx$$

$\int_a^b v(x)dx > 0$

$$\Rightarrow \quad m \leq \frac{\int_a^b u(x) v(x) dx}{\int_a^b v(x)\, dx} \leq M$$

$m = u(x_m)$, $x_m \in [a,b]$

$M = u(x_M)$, $x_M \in [a,b]$

Since $u \in C[a,b]$, by the intermediate value thm,

$\exists\ c \in [a,b]$ s.t.

$$u(c) = \frac{\int_a^b u(x)v(x)\,dx}{\int_a^b v(x)\,dx}$$

$$\Rightarrow \int_a^b u(x)v(x)\,dx = u(c)\cdot \int_a^b v(x)\,dx$$

#

Apply 2nd MVT to

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)\cdot (x-t)^n\,dt$$

with

$$u(t) = \begin{cases} f^{(n+1)}(t) & \text{if } x \geq 0 \\ (-1)^n f^{(n+1)}(t) & \end{cases}$$

$$v(t) = \begin{cases} \frac{(x-t)^n}{n!} & \text{if } x \geq 0 \\ \frac{(t-x)^n}{n!} & \text{if } x < 0 \end{cases}$$

We get:

$$R_n(x) = \int_0^x u(t)\, v(t)\, dt \qquad (v(t) \geq 0)$$

$\exists c \in [a,b]$ s.t.

$$\underset{\text{2nd MVT}}{=} u(c) \cdot \int_0^x v(t)\, dt \qquad \frac{d}{dt}\left(-\frac{(x-t)^{n+1}}{(n+1)!}\right)$$

$$= \begin{cases} f^{(n+1)}(c) \cdot \int_0^x \frac{(x-t)^n}{n!} dt & \text{if } x \geq 0 \\ = f^{(n+1)}(c) \cdot \left(-\frac{(x-t)^{n+1}}{(n+1)!}\right)\Big|_{t=0}^{x} \\ = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot x^{n+1} \\ & \frac{d}{dt}\left(\frac{(t-x)^{n+1}}{(n+1)!}\right) \\ (-1)^n f^{(n+1)}(c) \cdot \int_0^x \frac{(t-x)^n}{n!} dt & \text{if } x < 0 \end{cases}$$

$$= (-1)^n f^{(n+1)}(c) \cdot \frac{(t-x)^{n+1}}{(n+1)!}\Big|_{t=0}^{x}$$

$$= (-1)^n f^{(n+1)}(c) \left(0 - \frac{(-x)^{n+1}}{(n+1)!}\right)$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

Cor

$$R_n(x) \overset{\text{def.}}{=} \frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x-t)^n\, dt$$

$$\overset{\text{Cor}}{=} \frac{f^{(n+1)}(c)}{(n+1)!} \cdot x^{n+1} \quad \text{for some } c \text{ between } 0 \text{ and } x$$

$\Rightarrow$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \underbrace{\boxed{\frac{f^{(n+1)}(c)}{(n+1)!}\cdot x^{n+1}}}_{}$$

$R_n(x)$ $=$

for some c $(= c(x))$ between 0 and x

<u>Remark</u> (Answer to fundamental questions) $Q(ii')$

① $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(x)$

$$\Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$$

② To study $\lim_{n\to\infty} R_n(x)$, note that

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}\cdot x^{n+1} \right|$$

$$\leq \max_{t\in J_x} |f^{(n+1)}(t)| \cdot \frac{x^{n+1}}{(n+1)!}$$

where J_x is the closed interval that joins 0 and x