Calculus, Spring 2026, week 2

Recall

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k \right)$$

$$= \lim_{n\to\infty} \left(a_1 + a_2 + \cdots + a_n \right)$$

Example

Consider

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

$$= \lim_{n\to\infty} \left(\sum_{k=0}^{n} x^k \right)$$

Since

$$\sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n$$

$$-)\quad x \sum_{k=0}^{n} x^k = x + x^2 + \cdots + x^n + x^{n+1}$$

$k=0$

$$(1-x)\cdot\sum_{k=0}^{n} x^k = 1 - x^{n+1}$$

we have

$$\sum_{k=0}^{n} x^k \overset{\text{if } x\neq 1}{=} \frac{1-x^{n+1}}{1-x}$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x}$$

if $|x|<1$: $x^{n+1} \to 0$ as $n\to\infty$

if $|x|>1$ or $x=-1$: diverges

$$\sum_{k=0}^{\infty} x^k \overset{\text{if } |x|<1}{=} \begin{cases} \dfrac{1}{1-x} & \text{if } |x|<1 \\ \text{diverges} & \text{if } |x|>1 \text{ or } x=-1 \end{cases}$$

$= 1+1+1+\cdots$ if $x=1$

$= \lim_{n\to\infty}(n+1)$

diverges

Conclusion:

$$= \frac{1}{1-x} \quad \text{if } |x|<1$$

$$\sum_{k=0}^{\infty} x^k \begin{cases} \\ \text{diverges} & \text{if } |x| \geq 1 \end{cases}$$

#

Remark

Let $p \in \mathbb{N}$.

$$\sum_{k=0}^{\infty} a_k \text{ converges}$$

Note: $\sum_{k=0}^{\infty} a_k = \sum_{k=p}^{\infty} a_k + \sum_{k=0}^{p-1} a_k \in \mathbb{R}$

$$\Leftrightarrow \sum_{k=p}^{\infty} a_k \text{ converges}$$

Thm (Thm 12.2.4)

Let $a_k, b_k, \alpha, \beta \in \mathbb{R}$.

If $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ converges to L

$\sum_{k=1}^{\infty} b_k$ converges to $M = \lim_{n\to\infty} \sum_{k=1}^{n} b_k$

then

$$\sum_{k=1}^{\infty} (\alpha \cdot a_k + \beta \cdot b_k) \text{ converges}$$

to $\alpha \cdot L + \beta \cdot M$

$$\parallel$$

$$\alpha \cdot \lim_{n\to\infty} \sum_{k=1}^{n} a_k + \beta \cdot \lim_{n\to\infty} \sum_{k=1}^{n} b_k$$

$$\parallel$$

$$\lim_{n\to\infty} \left(\alpha \cdot \sum_{k=1}^{n} a_k + \beta \cdot \sum_{k=1}^{n} b_k \right)$$

$$\parallel$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\alpha \cdot a_k + \beta \cdot b_k)$$

$$\alpha(a_1 + a_2 + \cdots + a_n) + \beta(b_1 + b_2 + \cdots + b_n)$$

$$= \alpha a_1 + \alpha a_2 + \cdots + \alpha a_n + \beta b_1 + \cdots + \beta b_n$$

Example

① $\displaystyle\sum_{k=0}^{\infty} \left(\left(\frac{1}{2}\right)^k + 2 \cdot \left(\frac{1}{3}\right)^k \right)$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k + 2 \cdot \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$$

$$= \frac{1}{\quad} + 2 \cdot \frac{1}{\quad} = 2 + 2 \cdot \frac{3}{\quad}$$

$$1-\frac{1}{2} \qquad \qquad 1-\frac{1}{3} \qquad \qquad 2$$

$$= S \quad \#$$

② Consider $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$

Note that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$

n times

$$\geq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}$$

$$= \frac{n}{\sqrt{n}} = \sqrt{n} \to +\infty \quad \text{as } n\to\infty$$

$\Rightarrow \displaystyle\left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right)_{n=1}^{\infty}$ is unbounded

$\Rightarrow$ it is divergent.

$\Rightarrow \displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges. #

Fundamental question about series:

Is $\sum_{k=1}^{\infty} a_k$ Convergent?

<u>Thm</u> (n^{th}-term Test)

$$\sum_{k=1}^{\infty} a_k \text{ Converges} \Rightarrow \lim_{k\to\infty} a_k = 0$$

Or equivalently,

$$\lim_{k\to\infty} a_k \neq 0 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

<u>pf</u>

Let $S_n = \sum_{k=1}^{n} a_k$

If $\sum_{k=1}^{\infty} a_k$ converges, then $\exists L \in \mathbb{R}$

s.t.

$$\lim_{n\to\infty} S_n = L$$

Since

$$\lim_{n\to\infty} S_{n-1} = L$$

we have

$$0 = L - L = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= \lim_{n\to\infty} \left(S_n - S_{n-1} \right)$$

$$= \lim_{n\to\infty} \left(\underbrace{\sum_{k=1}^{n} a_k}_{(a_1+\cdots)+a_n} - \underbrace{\sum_{k=1}^{n-1} a_k}_{a_1+\cdots+a_{n-1}} \right)$$

$$= \lim_{n\to\infty} a_n = 0 \qquad \#$$

Example

① $\displaystyle\sum_{k=0}^{\infty} \frac{k}{k+1}$ diverges, since

$$\lim_{k\to\infty} \frac{k \times \frac{1}{k}}{(k+1)\times\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1 \neq 0$$

② $\displaystyle\sum_{k=1}^{\infty} \cos\left(\frac{1}{k}\right)$ diverges, since

$\cos x$ is continuous

$$\lim \cos\left(\frac{1}{k}\right) \overset{\downarrow}{=} \cos\left(\lim_{k\to\infty} \frac{1}{k}\right) = \cos 0$$

$k\to\infty$ $= 1 \neq 0$

Now we consider

$$\sum_{k=1}^{\infty} a_k \quad \text{with} \quad a_k \geq 0$$

$$\Rightarrow \left(\sum_{k=1}^{n} a_k\right)_{n=1}^{\infty} \text{ is increasing}$$

<u>Thm</u> (Thm 12.3.1)

A series $\sum_{k=1}^{\infty} a_k$ with non-negative terms converges

$$\Leftrightarrow \left(\sum_{k=1}^{n} a_k\right)_{n=1}^{\infty} \text{ is bounded}$$

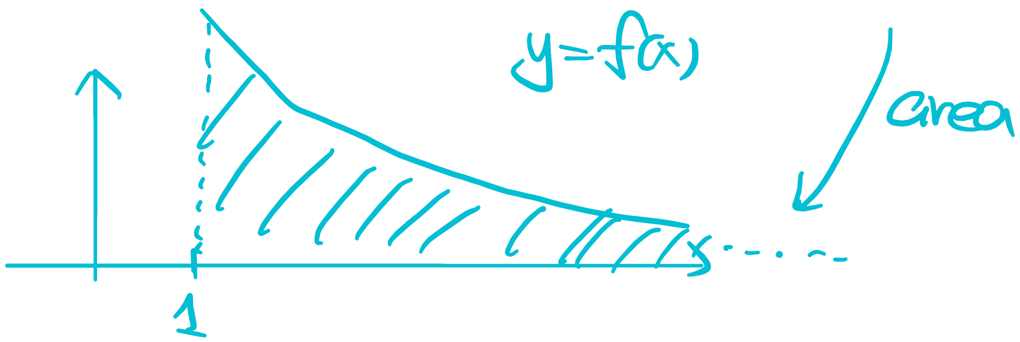
<u>Thm</u> (Integral Test, Thm 12.3.2)

If f is continuous, positive and decreasing on $[1, \infty)$, then

$$\sum_{k=1}^{\infty} f(k) \quad \text{Converges}$$

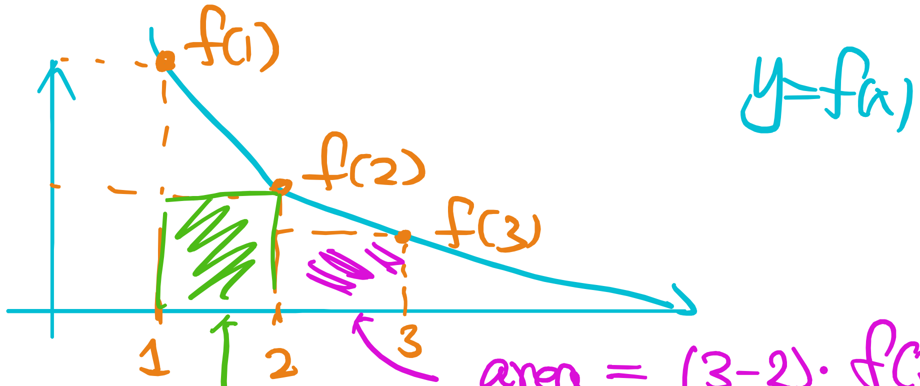
$$\Longleftrightarrow \quad \int_1^{\infty} f(x)\,dx \quad \text{Converges}$$

Recall: $\displaystyle\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^{b} f(x)\,dx$



Pf

Note that



area $= (3-2)\cdot f(3) = f(3)$

area $= (2-1)\cdot f(2) = f(2)$

$\Rightarrow$

$$\sum_{k=2} f(k) \leq \int_1 f(x)\,dx$$

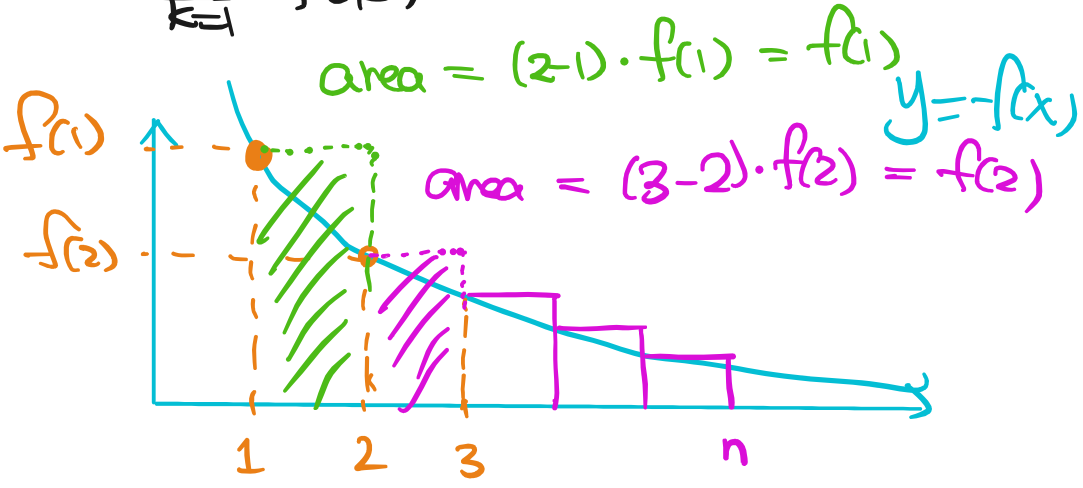
So if $\int_1^\infty f(x)\,dx$ converges, then

$$\sum_{k=2}^{n} f(k) \underset{\substack{\uparrow \\ \because f(x) > 0}}{\leq} \int_1^\infty f(x)\,dx \qquad \forall n \geq 2,\ n \in \mathbb{N}$$

$\Rightarrow$ $\sum_{k=2}^{\infty} f(k)$ converges $\Rightarrow$ $\sum_{k=1}^{\infty} f(k)$ converges.

Conversely if $\sum_{k=1}^{\infty} f(k)$ converges, then

$\sum_{k=1}^{n-1} f(k)$ is bounded



$$\sum_{k=1}^{\infty} f(k) \geq \sum_{k=1}^{n-1} f(k) \geq \int_1^n f(x)\,dx$$

$$\Rightarrow \int_1^b f(x)\,dx \underset{}{\overset{f(x)>0}{\leq}} \int_1^{[b]+1} f(x)\,dx \leq \sum_{k=1}^{\infty} f(k)$$

as $b \to \infty$

increasing, bounded above

$$\Rightarrow \int_1^\infty f(x)\,dx \quad \text{converges.} \qquad \#$$

<u>Example</u> (p-series)

Consider

$$f(x) = \frac{1}{x}$$

which is continuous, positive, decreasing on $[1, \infty)$.

Since

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \left(\ln x \Big|_1^b \right) = \lim_{b\to\infty} \ln b$$

diverges, by the integral test, we know

$$\sum^\infty \frac{1}{} \quad \text{diverges}$$

$\sum_{k=1} \frac{\;}{k}$ converges.

More generally, since

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx \qquad \frac{1}{x^p} = x^{-p}$$

$$= \lim_{b\to\infty}\begin{cases} \ln b & p=1 \\ \boxed{\left.\dfrac{x^{-p+1}}{-p+1}\right|_{x=1}^{b}} & p\neq 1 \end{cases}$$

$$= \frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

$$\begin{cases} \text{Converges} & -p+1<0 \Leftrightarrow p>1 \\ \text{diverges} & p\leq 1 \end{cases}$$

we know that

$$\sum_{k=1}^{\infty}\frac{1}{k^p}\begin{cases} \text{Converges} & \text{if } p>1 \\ \text{diverges} & \text{if } p\leq 1 \end{cases}$$

#

e.g. $\sum_{k=1}^{\infty} \frac{1}{k^2}$, $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converge

$\sum_{k=1}^{\infty} \frac{1}{k}$, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverge

<u>Thm</u> (Basic Comparison Test, Thm 12.3.6)

Let $a_k, b_k \geq 0$. Suppose

$\exists N$ s.t. $\forall k \geq N$, $a_k \leq b_k$

indep of k

$$a_k \leq b_k \quad \text{for } k \text{ sufficiently large}$$

$$\left(\Rightarrow \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k \right)$$

(i) $\sum_{k=1}^{\infty} b_k$ converges $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ converges

(ii) $\sum_{k=1}^{\infty} a_k$ diverges $\Rightarrow$ $\sum_{k=1}^{\infty} b_k$ diverges

<u>Example</u>

① $\sum_{k=1}^{\infty} \frac{1}{2k^3+1}$ converges because

$$\frac{1}{2k^3+1} < \frac{1}{k^3} \quad \forall k \geq 1$$

$k^3 + (k^3+1) > k^3$

and $\sum_{k=1}^{\infty} \frac{1}{k^3}$ Converges

#

② $\sum_{k=1}^{\infty} \frac{1}{3k+1}$ diverges because

$$\frac{1}{3k+1} \geqslant \frac{1}{3k+k} = \frac{1}{4k} \quad \forall k \geq 1$$

and $\sum_{k=1}^{\infty} \frac{1}{4k} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

#

Thm (Limit Comparison Test, Thm 12.3.7)

Let $a_k, b_k > 0$. If

$$\lim_{k\to\infty} \frac{a_k}{b_k} \neq 0,$$

then

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \sum_{k=1}^{\infty} b_k \text{ converges.}$$

Pf

pf

Let $L = \lim_{k\to\infty} \frac{a_k}{b_k} > 0.$

For $\varepsilon = \frac{L}{2} > 0$, $\exists\, N = N_\varepsilon$ s.t.

$$\left| \frac{a_k}{b_k} - L \right| < \frac{L}{2} \quad \forall\, k \geq N.$$

$\Leftrightarrow$

$$\underset{\frac{L}{2}}{\underset{\shortparallel}{L - \frac{L}{2}}} < \frac{a_k}{b_k} < L + \frac{L}{2} = \frac{3}{2}L$$

$b_k > 0$

$$\Rightarrow \quad \frac{L}{2} b_k < a_k < \frac{3}{2} L b_k \quad \forall\, k \geq N$$

By Basic Comparison Test,

(i) $\sum_{k=1}^{\infty} a_k$ Converges $\Rightarrow \sum_{k=1}^{\infty} \frac{L}{2} b_k$ Converges

$$\Rightarrow \frac{2}{L} \cdot \sum_{k=1}^{\infty} \frac{L}{2} b_k = \sum_{k=1}^{\infty} b_k \text{ Converges.}$$

(ii) $\sum_{k=1}^{\infty} b_k$ Converges $\Rightarrow \sum_{k=1}^{\infty} \frac{3}{2} L\, b_k$ Converges

$$\Rightarrow \sum_{k=1}^{\infty} a_k \quad \text{converges.} \qquad \#$$

<u>Example</u>

① $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k - 3}$ converges because

$$\lim_{k\to\infty} \frac{\frac{1}{5^k-3}}{\frac{1}{5^k}} = \lim_{k\to\infty} \frac{1}{1-\frac{3}{5^k}} = 1 \neq 0$$

$\frac{3}{5^k} \to 0$

and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k}$ converges $\#$

② $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{\sqrt{k^3+1}}$ diverges because

$$\lim_{k\to\infty} \frac{\frac{(2k+1)/k}{(\sqrt{k^3+1})/k}}{\frac{1}{\sqrt{k}}} = \lim_{k\to\infty} \frac{2+\frac{1}{k}}{\sqrt{k+\frac{1}{k^2}} \cdot \frac{1}{\sqrt{k}}}$$

$$= \lim_{k\to\infty} \frac{2+\frac{1}{k}}{\sqrt{1+\frac{1}{k^3}}} = 2 \neq 0$$

$\frac{1}{k} \to 0$, $\frac{1}{k^3} \to 0$

and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges #

③ $\displaystyle\sum_{k=1}^{\infty} \frac{2k+5}{\sqrt{k^6+3k^3}}$ Converges because

$$\lim_{k\to\infty} \frac{\frac{2k+5}{\sqrt{k^6+3k^3}}}{\frac{1}{k^2}} = 2 \neq 0$$

and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ Converges. #

<u>Thm</u> (Root Test, Thm 12.4.1)

Let $a_k \geq 0$. Suppose

$$\rho = \overline{\lim_{k\to\infty}} \, a_k^{\frac{1}{k}} \quad \left(= \lim_{k\to\infty} a_k^{\frac{1}{k}} \text{ if the limit exists} \right)$$

(i) If $\rho < 1$, then $\displaystyle\sum_{k=1}^{\infty} a_k$ Converges

(ii) If $\rho > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(iii) If $\rho = 1$, no conclusion

e.g. $\sum \frac{1}{k}$ diverges, $\rho = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} = 1$

$\sum \frac{1}{k^2}$ Converges, $\rho = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k^2}} = 1$

Pf

Recall that if $\overline{\lim_{k\to\infty}}\, b_k = \beta$, then

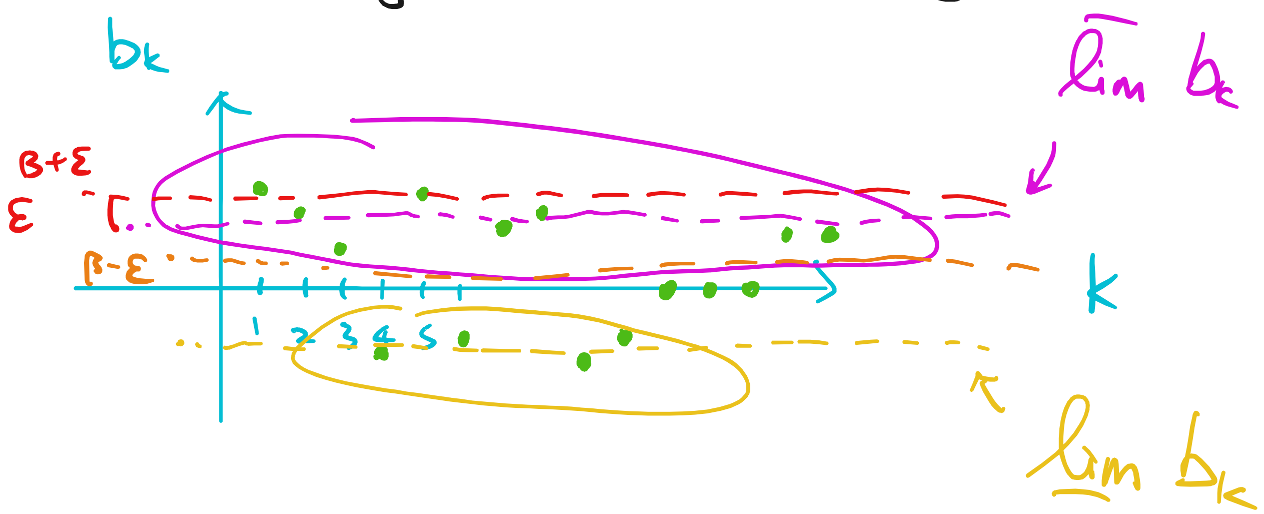
$\forall \varepsilon > 0$, $\exists N = N_\varepsilon$ s.t.

(i) $b_k < \beta + \varepsilon \quad \forall k \geq N$

(ii) $\exists$ subsequence $(b_{k_j})_{j=1}^{\infty}$ of $(b_k)_{k=1}^{\infty}$

s.t. $\left(|b_{k_j} - \beta| < \varepsilon \iff \beta - \varepsilon < b_{k_j} < \beta + \varepsilon \right)$

$b_{k_j} > \beta - \varepsilon \quad \forall j \geq 1$



If $\rho = \overline{\lim_{k\to\infty}} \sqrt[k]{a_k} < 1$, choose $\varepsilon = \frac{1-\rho}{2} > 0$

$$\Rightarrow \rho + \varepsilon = \frac{1+\rho}{2} < 1$$

By (i), $\exists\ N = N_\varepsilon$ s.t.

$$\sqrt[k]{a_k} < \rho + \varepsilon \qquad \forall\ k \geq N_\varepsilon$$

$$\Rightarrow 0 \leq a_k < (\rho+\varepsilon)^k \qquad \forall\ k \geq N_\varepsilon$$

Since $\sum (\rho+\varepsilon)^k$ converges, by the basic comparison test, we know $\sum_{k=1}^{\infty} a_k$ converges.

If $\rho = \overline{\lim_{k\to\infty}} \sqrt[k]{a_k} > 1$, then choose

$$\varepsilon' = \frac{\rho-1}{2} > 0.$$

By (ii), $\exists$ subseq. $(a_{k_j})_j$ of $(a_k)_k$ s.t.

$$(a_{k_j})^{1/k_j} > \rho - \varepsilon' = \frac{\rho+1}{2} > 1$$

$\forall\ j \geq 1$

$k_j \geq j$

$$\Rightarrow a_{k_j} > \left(\frac{\rho+1}{2}\right)^{k_j} \geq \left(\frac{\rho+1}{2}\right)^{j} \to \infty$$

as $j \to \infty$

$\Rightarrow (a_{k_j})_j$ is unbounded

$\Rightarrow (a_k)_k$ is unbounded

$\Rightarrow \lim_{k\to\infty} a_k \neq 0$

By n^{th}-term test, $\sum_{k=1}^{\infty} a_k$ diverges. #

<u>Example</u>

① $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^k}$ Converges because

$$\overline{\lim_{k\to\infty}} \sqrt[k]{\frac{1}{(\ln k)^k}} = \overline{\lim_{k\to\infty}} \frac{1}{\ln k} = \underset{1}{\underset{\wedge}{0}}$$

(Root Test)

② $\sum_{k=1}^{\infty} \frac{2^k}{k^3}$ diverges because

$$\overline{\lim_{k\to\infty}} \sqrt[k]{\frac{2^k}{k^3}} = \overline{\lim_{k\to\infty}} \frac{2}{(\sqrt[k]{k})^3} = 2 > 1$$

1

(Root Test)

③ $\sum_{k=1}^{\infty} a_k$, where $a_k = \begin{cases} \left(\frac{1}{2}\right)^m & \text{if } k = 2m-1, \end{cases}$

$\left(\frac{1}{3}\right)^m$ if $k=2m$.

$$\Rightarrow \sqrt[k]{a_k} = \begin{cases} \sqrt[2m-1]{\left(\frac{1}{2}\right)^m} = \left(\frac{1}{2}\right)^{\frac{m}{2m-1}} \xrightarrow{\text{as } k\to\infty} \left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}} & k=2m-1 \\ \sqrt[2m]{\left(\frac{1}{3}\right)^m} = \sqrt{\frac{1}{3}} \to \frac{1}{\sqrt{3}} \text{ as } k\to\infty & k=2m \end{cases}$$

divergent

$(k\to\infty \Leftrightarrow m\to\infty)$

$\frac{1}{\sqrt{2}} \neq \frac{1}{\sqrt{3}}$

$$\rho = \overline{\lim_{k\to\infty}} \sqrt[k]{a_k} = \frac{1}{\sqrt{2}} < 1$$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ is Convergent (Root Test) #

Thm (Ratio Test, Thm 12.4.2)

Let $a_k > 0$. Suppose

$$\lambda = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

(i) $\lambda < 1 \Rightarrow \sum a_k$ converges

(ii) $\lambda > 1 \Rightarrow \sum a_k$ diverges

(iii) $\lambda = 1 \Rightarrow$ No conclusion

e.g. $\sum \frac{1}{k}$ div., $\lambda = \lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = 1$

$$\sum \frac{1}{k^2} \text{, conv. } \lambda = \lim_{k\to\infty} \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = 1$$

pf

By Root Test, it suffices to show that

$$\lambda = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} \Rightarrow \lim_{k\to\infty} \sqrt[k]{a_k} = \lambda$$

Let $\varepsilon > 0$ be an arbitrary positive number.

Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda$, $\exists\, N = N_\varepsilon \in \mathbb{N}$ s.t.

$$(*) \quad \lambda - \varepsilon < \frac{a_{k+1}}{a_k} < \lambda + \varepsilon \qquad \forall k \geq N$$

For $n > N$,

$$a_n = \underbrace{\overbrace{\frac{a_n}{a_{n-1}}}^{<\ \lambda+\varepsilon}}_{\lambda-\varepsilon\ <} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots\cdots \frac{a_{N+1}}{a_N} \cdot a_N$$

$(*) \Rightarrow$

if $\lambda > 0$, $\lambda > \varepsilon > 0$

$$a_N(\lambda-\varepsilon)^{n-N} < a_n < a_N(\lambda+\varepsilon)^{n-N}$$

if $\lambda = 0$

$$0 < a_n < a_N(\lambda+\varepsilon)^{n-N}$$

$$\Rightarrow \sqrt[n]{a_N}\,(\lambda-\varepsilon)^{\frac{n-N}{n}} < \sqrt[n]{a_n} < \sqrt[n]{a_N}\,(\lambda+\varepsilon)^{\frac{n-N}{n}}$$

$\frac{n-N}{n} \to 1$ as $n\to\infty$

$\Rightarrow$

1 as $n\to\infty$ $\qquad$ $a_N>0$, indep of n

$\lim\limits_{n\to\infty} (a_N)^{\frac{1}{n}} = (a_N)^0 = 1$

$$\underbrace{\lim_{n\to\infty} \sqrt[n]{a_N}\,(\lambda-\varepsilon)^{\frac{n-N}{n}}} \le \underline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}$$

$$\lim_{n\to\infty} (\lambda-\varepsilon)^{\frac{n-N}{n}} \le \overline{\lim_{n\to\infty}} \sqrt[n]{a_N}\,(\lambda+\varepsilon)^{\frac{n-N}{n}}$$

$$= \lambda-\varepsilon \qquad\qquad = \lambda+\varepsilon$$

So

$$\lambda-\varepsilon \le \underline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \lambda+\varepsilon$$

$\lambda>\varepsilon>0$ $\quad 0<\varepsilon<\lambda$

$\forall \varepsilon>0$

$$\Rightarrow \underline{\lim_{n\to\infty}} \sqrt[n]{a_n} = \overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = \lambda$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = \lambda \qquad \#$$

Example

① $\sum\limits_{k=1}^{\infty} \frac{1}{k!}$ Converges because

(Ratio Test)

$$\lim_{k\to\infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1$$

② $\sum_{k=1}^{\infty} \frac{k}{10^k}$ Converges because

(Ratio Test)

$$\lim_{k\to\infty} \frac{\frac{k+1}{10^{k+1}}}{\frac{k}{10^k}} = \lim_{k\to\infty} \frac{\frac{k+1}{k}}{10} = \frac{1}{10} < 1$$

③ $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ diverges because

(Ratio Test)

$$\lim_{k\to\infty} \frac{\frac{(k+1)^{k+\cancel{1}}}{\cancel{(k+1)!}\ \cancel{k+1}}}{\frac{k^k}{\cancel{k!}}} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k$$

$$= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e = 2.\cdots\cdots > 1$$

Thm (Generalized Ratio Test)

Let $a_k > 0$.

(i) $\overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ converges

(ii) $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ diverges

(iii) $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} \leq 1 \leq \overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k}$ (e.g.

$(\leq \varepsilon, \leq \frac{?}{k})$

$\Rightarrow$ No conclusion

pf

(i) Suppose $\overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} = \lambda < 1$.

For $\varepsilon = \frac{1-\lambda}{2} > 0$, $\exists\, N = N_\varepsilon \in \mathbb{N}$ s.t.

$\forall k \geq N$, $\quad \frac{a_{k+1}}{a_k} < \lambda + \varepsilon = \overset{\text{"}r\text{"}}{\frac{1+\lambda}{2}} < 1$

$\Rightarrow$

$r a_N > a_{N+1}$, $\quad \underbrace{r a_{N+1}}_{\wedge\ r^2 a_N} > a_{N+2}$, $\cdots$

$\Rightarrow \quad a_N + a_{N+1} + a_{N+2} + \cdots + a_n \leftarrow$ bounded

$< a_N + r a_N + r^2 a_N + \cdots + r^{n-N} a_N$

$= a_N \cdot \frac{1 - r^{(n-N+1)}}{1-r} < \frac{a_N}{1-r}$

$\Rightarrow \sum_{k=N}^{\infty} a_k$ Converges $\Rightarrow \sum_{k=1}^{\infty} a_k$ Converges.

(ii) $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \mu > 1$.

For $\varepsilon' = \frac{\mu-1}{2} > 0$, $\exists\, N' = N_{\varepsilon'} \in \mathbb{N}$ s.t.

$$\frac{a_{k+1}}{a_k} > \mu - \varepsilon = \frac{\mu+1}{2} > 1 \quad \forall k \geq N'$$

$\Rightarrow \quad a_{k+1} > a_k > \cdots > a_{N'} > 0 \quad \forall k \geq N'$

$$\Rightarrow \quad \lim_{k\to\infty} a_k \neq 0 \quad \Rightarrow \quad \sum_{k=1}^{\infty} a_k \text{ diverges. } \#$$

Remark

In fact, we have (assume $a_k > 0$)

$$\underline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} \leq \underline{\lim_{k\to\infty}} \sqrt[k]{a_k} \leq \overline{\lim_{k\to\infty}} \sqrt[k]{a_k} \leq \overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k}$$

Example

Consider

$$a_k = \begin{cases} \left(\frac{1}{2}\right)^k & \text{if } k \text{ is even} \\ \left(\frac{1}{3}\right)^k & \text{if } k \text{ is odd.} \end{cases}$$

The series is

$$\sum_{k=0}^{\infty} a_k = 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{27} + \cdots$$

① Ratio Test:

For $k = 2m$,

$$\frac{a_{k+1}}{a_k} = \frac{a_{2m+1}}{a_{2m}} = \frac{\left(\frac{1}{3}\right)^{2m+1}}{\left(\frac{1}{2}\right)^{2m}}$$

$$= \frac{1}{3}\left(\frac{2}{3}\right)^{2m} \longrightarrow 0 \quad \text{as } m \to \infty$$

For $k = 2m-1$,

$$\frac{a_{k+1}}{a_k} = \frac{a_{2m}}{a_{2m-1}} = \frac{\left(\frac{1}{2}\right)^{2m}}{\left(\frac{1}{3}\right)^{2m-1}}$$

$$= \frac{1}{2} \cdot \left(\frac{3}{2}\right)^{2m-1} \to \infty \quad \text{as } m \to \infty$$

$$\Rightarrow \quad \underline{\lim}_{k\to\infty} \frac{a_{k+1}}{a_k} = 0 < 1 < \overline{\lim}_{k\to\infty} \frac{a_{k+1}}{a_k} = \infty$$

$\Rightarrow$ NO conclusion from Ratio Test !!

② Root Test:

$$\sqrt[k]{a_k} = \begin{cases} \sqrt[k]{\left(\frac{1}{2}\right)^k} = \frac{1}{2} & k \text{ even} \\ \sqrt[k]{\left(\frac{1}{3}\right)^k} = \frac{1}{3} & k \text{ odd} \end{cases}$$

$$\Rightarrow \quad \overline{\lim}_{k\to\infty} \sqrt[k]{a_k} = \frac{1}{2} < 1$$

$\Rightarrow \sum_{k=0}^{\infty} a_k$ converges (by Root Test) #

<u>Def</u>

A series $\sum^{\infty} a_k$ is absolut[...]

A series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

A series is conditionally convergent if it convergent and not absolutely convergent.

Thm (Thm 12.5.1)
Absolutely convergent series are convergent.