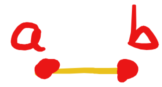
Calculus, Spring 2026, week 15

§ Green's Thm

Recall

$$\circledast \quad \int_a^b f'(x)\,dx = f(b) - f(a)$$

To obtain higher dim. version of this equation, observe that:



"1-dim. space"

- LHS = integration of f' over $[a,b]$
- RHS = integration of f over $\{a,b\}$ (0-dim. space)

$\{a,b\} = $ boundry of $[a,b]$

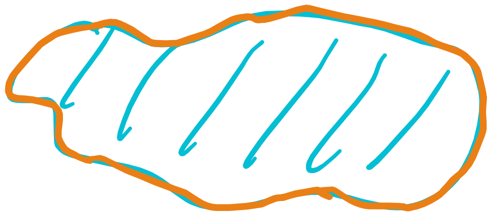
So $\circledast$ can be considered as a way to reduce a 1-dimensional integral to a 0-dimensional integral (over boundaries)

This observation leads us to see

Green's Thm: a 2-dimensional version of fundamental thm of Calculus $\circledast$

Since the boundary of 2-dim. region is generally a curve, we need: " integration over a curve "

Expect:



$$\iint \underline{(\text{some differential of } f)}\, dx\,dy \overset{??}{}$$

$$= \int f \, dx \quad ??$$

Def (Def 18.1.3)

Let

$$C: \quad \vec{\gamma}(t) = (\underset{x}{\gamma_1(t)}, \underset{y}{\gamma_2(t)}), \quad t \in [a,b]$$

be a smooth parametrized curve.

Suppose $P(x,y)$, $Q(x,y)$ are smooth functions on $\mathbb{R}^2$. The line integral

$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

over C is defined to be

$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

"$x = r_1(t)$"
$y = r_2(t)$
"$\Downarrow$"
"$dx = r_1'(t)dt$
$dy = r_2'(t)dt$"

$$= \int_a^b P(r_1(t), r_2(t))\, r_1'(t)dt + Q(r_1(t), r_2(t))\, r_2'(t)dt$$

definition

$$= \boxed{\int_a^b \left(P(r_1(t), r_2(t))r_1'(t) + Q(r_1(t), r_2(t))r_2'(t)\right)dt}$$

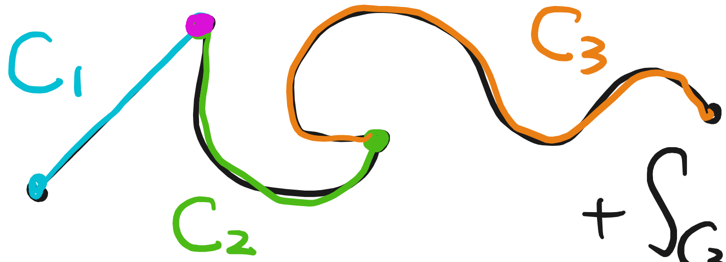
$$= \int_a^b \left(P(\vec{r}(t)), Q(\vec{r}(t))\right)\cdot \vec{r}\,'(t)\, dt$$

<u>Remark</u>

① The definition of line integrals naturally extends to piecewise smooth curves:

continuous, NOT differentiable

C:

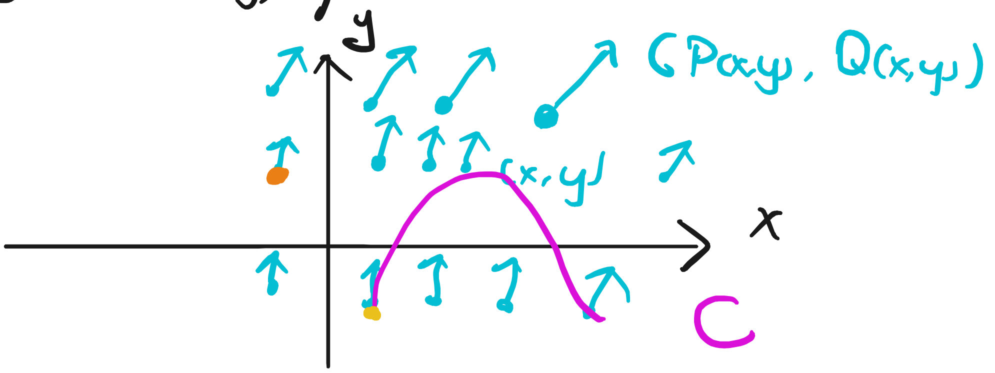


$$+\int_{C_3} Pdx + Qdy$$

$$\Rightarrow \int_C Pdx + Qdy = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy$$

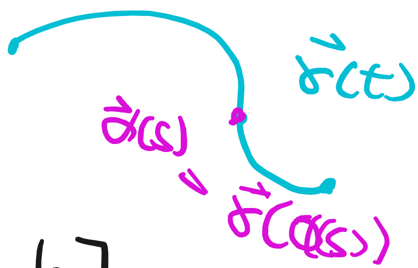
② $\int_C P\,dx + Q\,dy$ = "work" 功

$(P(x,y), Q(x,y))$ = "vector field"



<u>Thm</u> (Thm 18.1.3)

A line integral is <u>invariant</u> (不變的) under every direction-presering change of parameter.



That is, if

$$\phi : [c,d] \to [a,b]$$

is an increasing smooth function s.t.

$$\phi'(s) > 0 \quad \forall\ s \in (c,d)$$

and

another parametrization of C with same direction

$$\vec{\alpha}(s) := \vec{\gamma}(\phi(s)), \quad s \in [c,d]$$

then $\alpha_1(s) = \gamma_1(\phi(s))$, $\alpha_2(s) = \gamma_2(\phi(s))$

$$\int_a^b \left(P(\vec{\gamma}(t)), Q(\vec{\gamma}(t))\right) \bullet \vec{\gamma}'(t)\, dt$$

$$= \int_c^d \left(P(\vec{\alpha}(s)), Q(\vec{\alpha}(s))\right) \bullet \vec{\alpha}'(s)\, ds$$

Pf

$$\int_{a_1}^{a_2} f(u(t)) \cdot u'(t)\, dt = \int_{u(a_1)}^{u(a_2)} f(u)\, du$$

$$\text{LHS} = \int_a^b P(\gamma_1(t), \gamma_2(t))\, \gamma_1'(t) + Q(\gamma_1(t), \gamma_2(t))\, \gamma_2'(t)\, dt$$

$s = \phi^{-1}(t)$

$t = \phi(s)$

$\Rightarrow dt = \phi'(s) ds$

$$= \int_{\underbrace{\phi^{-1}(a)}_{=c}}^{\underbrace{\phi^{-1}(b)}_{=d}} \left(\begin{array}{l} P(\underbrace{\gamma_1(\phi(s))}_{=\alpha_1(s)}, \underbrace{\gamma_2(\phi(s))}_{=\alpha_2(s)})\, \gamma_1'(\phi(s)) \\ + Q(\underbrace{\gamma_1(\phi(s))}_{=\alpha_1(s)}, \underbrace{\gamma_2(\phi(s))}_{=\alpha_2(s)})\, \gamma_2'(\phi(s)) \end{array} \right) \phi'(s)\, ds$$

chain rule $(\gamma_1(\phi(s)))' = \alpha_1'(s)$

chain rule $(\gamma_2(\phi(s)))' = \alpha_2'(s)$

$$= \int_c^d \left(\begin{array}{l} P(\alpha_1(s), \alpha_2(s))\, \alpha_1'(s) \\ + Q(\alpha_1(s), \alpha_2(s))\, \alpha_2'(s) \end{array} \right) ds$$

$= \text{RHS}$ #

Similarly, we have:

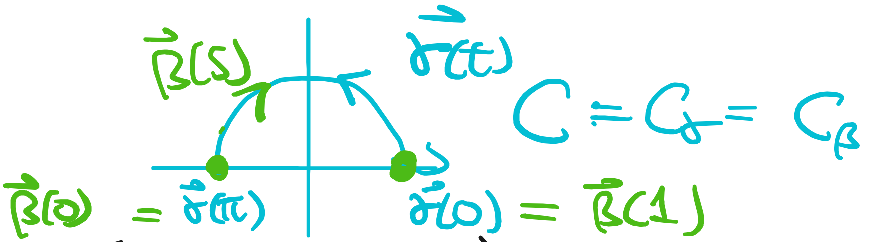
Thm

If $\psi : [c,d] \to [a,b]$ is a decreasing smooth function s.t. $\psi'(s) < 0 \ \forall s \in (c,d)$, and $\vec{\beta}(s) := \vec{\gamma}(\psi(s))$, $s \in [c,d]$,

another parametriation with the reversed direction

then
$$\int_a^b \left(P(\vec{\gamma}(t)), Q(\vec{\gamma}(t))\right) \bullet \vec{\gamma}'(t)\,dt$$
$$= -\int_c^d \left(P(\vec{\beta}(s)), Q(\vec{\beta}(s))\right) \bullet \vec{\beta}'(s)\,ds$$

Example



Let
$$\vec{\gamma}(t) = (\underbrace{\cos t}_{= x}, \underbrace{\sin t}_{= y}), \quad t \in [0, \pi]$$
$$\vec{\beta}(s) = (\cos\boxed{\pi - s}, \sin(\pi - s)), \quad s \in [0,\pi]$$
$\boxed{\pi - s} = \psi(s)$

$$\Rightarrow_{①} \int_{C_\gamma} y\,dx - x\,dy \qquad \begin{pmatrix} dx = -\sin t\,dt \\ dy = \cos t\,dt \end{pmatrix}$$
$$= \int_0^\pi \sin t\,(-\sin t)\,dt - \cos t \cdot \cos t\,dt$$
$$= \int_0^\pi \underbrace{(-\sin^2 t - \cos^2 t)}_{= -1}\,dt = -\pi$$

② $\int_{C_3} y\,dx - x\,dy$

$x = \cos(\pi - s) = -\cos s$

$y = \sin(\pi - s) = \sin s$

$\Rightarrow dx = \sin s\, ds$

$dy = \cos s\, ds$

$$= \int_0^{\pi} \sin s \cdot \sin s\, ds - (-\cos s) \cos s\, ds$$

$$= \int_0^{\pi} (\underbrace{\sin^2 s + \cos^2 s}_{=1})\, ds = \pi \quad \#$$

<u>Def</u>

A (parametrized) curve in $\mathbb{R}^2$

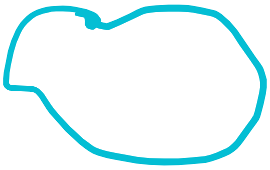
$$C: \vec{\gamma} = \vec{\gamma}(t), \quad t \in [a,b],$$

is called <u>closed</u> if $\vec{\gamma}(a) = \vec{\gamma}(b)$

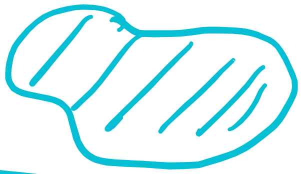
and called <u>simple</u> if it doesn't intersect itself: $t_1 \neq t_2 \in (a,b) \Rightarrow \vec{\gamma}(t_1) \neq \vec{\gamma}(t_2)$

<u>Jordan curve</u> = closed and simple

A closed region Ω is a <u>Jordan region</u> if its total boundary is a Jordan curve.



Jordan curve



Jordan region

closed, not simple

NOT Jordan region

simple, not closed

Remark

If C is a closed curve, we usually write ($\oint_C Pdx+Qdy$)

$$\oint_C Pdx+Qdx = \int_C Pdx+Qdx$$

(clockwise)

if the parametrization is counterclockwise

Thm (Green's Thm, Thm 18.5.1)

Let Ω be a Jordan region with a piecewise smooth boundary C.

If P and Q are smooth functions on Ω then

$\int_a^b f'(x)\,dx$

"exterior derivative" of $P\,dx + Q\,dy$

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y) \right) dx\,dy$$

$f(b) - f(a)$

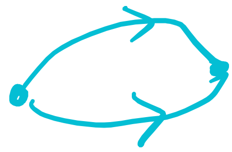
$$= \oint_C P(x,y)\,dx + Q(x,y)\,dy$$

Example

$$\oint_C \overset{P}{\overset{\shortparallel}{\underline{e^x \sin y}}}\,dx + \overset{Q}{\overset{\shortparallel}{\underline{e^x \cos y}}}\,dy = ?$$

$$C = \{x^2 + y^2 = 1\}$$

Sol



By Green's Thm,

$$\text{ans} = \iint_{\substack{\Omega \\ \{x^2+y^2 \le 1\}}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

Since

$$\frac{\partial Q}{\partial x} = e^x \cos y \quad , \quad \frac{\partial P}{\partial y} = e^x \cos y$$

we have

$$= \iint_{\Omega} 0 \, dx\,dy = 0$$

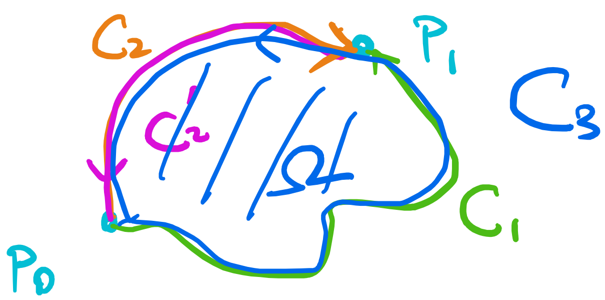
#

Remark

If $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$ on $\mathbb{R}^2$, then

$$\int_C P\,dx + Q\,dy$$

"only depends" on the starting point and end point of C:



$$\Rightarrow \oint_{C_3} P\,dx + Q\,dy = \iint_{\Omega} (Q_x - P_y)\,dx\,dy$$

$$\Vert \qquad\qquad\qquad = 0$$

$$\int_{C_1} P\,dx + Q\,dy + \int_{C_2'} P\,dx + Q\,dy$$

$$= \int_{C_1} \boxed{} - \int_{C_2} \boxed{}$$

$$\Rightarrow \int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy$$

<u>Example</u>

$$\oint_C \overbrace{\underline{(3x^2+y)}}^{P}\,dx + \overbrace{\underline{(2x+y^3)}}^{Q}\,dy = ?$$

C

where $C = \{ x^2+y^2 = 3^2 \}$

<u>sol</u>

$$\boxed{Q_x - P_y = 2 - 1 = 1}$$

<u>Green's Thm</u>

$$\underline{\text{ans}} = \iint_{\Omega} 1 \; dxdy = 9\pi \quad \#$$

<u>Example</u>

Let C be a Jordan curve that does not pass through $\vec{O} = (0,0)$.

Show that

"Cauchy integral formula"

$$\oint_C \overbrace{-\frac{y}{\underset{\sim\sim\sim}{x^2+y^2}}}^{P \,=}\,dx + \overbrace{\frac{x}{\underset{\sim\sim\sim}{x^2+y^2}}}^{Q\,=}\,dy$$

NOT smooth at $\vec{O}$

$$= \begin{cases} 0 & \text{if } C \text{ does not enclose the origin.} \\ 2\pi & \text{if } C \text{ does enclose the origin.} \end{cases}$$

pf

Case 1 C doesn't enclose $\vec{0}$

Apply Green's Thm: P, Q are smooth in

$$Q_x - P_y$$

$$= \left(\frac{x}{x^2+y^2}\right)_x - \left(-\frac{y}{x^2+y^2}\right)_y$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$= \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{1(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$-2(x^2+y^2)$$

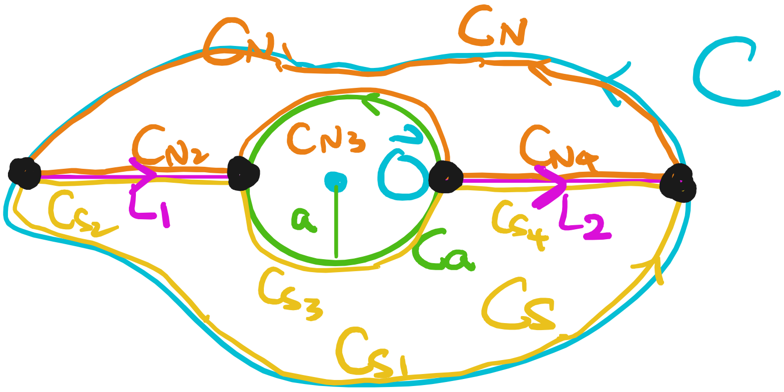
$$= 0 \quad \forall (x,y) \neq (0,0).$$

Green's $\Rightarrow$

$$\oint_C P\,dx + Q\,dy = \iint_\Omega Q_x - P_y \, dx\,dy$$

$$= 0$$

Case 2: C does enclose $\vec{0}$



A geometric fact: $\exists$ a circle

$$C_a: \quad x^2+y^2=a^2$$

inside C. Consider the auxiliary curves

Note that

① C_N and C_S do NOT enclose $\vec{0}$

$$\overset{\text{case1}}{\Longrightarrow} \quad \oint_{C_N} \underline{Pdx+Qdy} = \oint_{C_S} Pdx+Qdy = 0$$

② $$\oint_{C_N} \square = \int_{C_{N_1}} + \overset{=\int_{L_1}}{\int_{C_{N_2}}} + \int_{C_{N_3}} + \overset{\int_{L_2}\,=}{\int_{C_{N_4}}}$$

$$= \int_{C_{N_1}} + \int_{L_1} + \int_{C_{N_3}} + \int_{L_2}$$

③ $$\oint_{C_S} = \int_{C_{S_1}} + \int_{C_{S_2}} + \int_{C_{S_3}} + \int_{C_{S_4}}$$

$$\int_{C_{S_2}} = -\int_{L_1} \qquad \int_{C_{S_4}} = -\int_{L_2}$$

④ ② + ③ :

$$\oint_{C_N} + \oint_{C_S} \overset{①}{=} 0$$

$$= \int_{C_{N_1}} + \int_{C_{N_3}} + \int_{C_{S_1}} + \int_{C_{S_3}}$$

$$= \oint_C \qquad - \oint_{C_a}$$

$$= \oint_C - \oint_{C_a} = 0$$

$$\Rightarrow \oint_C = \oint_{C_a}$$

C_a : $x^2+y^2=a^2$

$x = a\cos\theta$

$y = a\sin\theta$

$\theta \in [0, 2\pi]$

⑤ $$\text{ans} = \oint_{C_a} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$\overset{\text{def. of line integral}}{=} \int_0^{2\pi} -\frac{a\sin\theta}{\underbrace{(a\cos\theta)^2+(a\sin\theta)^2}_{=a^2}} \underbrace{a(-\sin\theta)\,d\theta}_{dx} + \frac{a\cos\theta}{(a\cos\theta)^2+(a\sin\theta)^2} \underbrace{a\cos\theta\,d\theta}_{dy}$$

$$= \int_0^{2\pi} \frac{\cancel{a^2\sin^2\theta + a^2\cos^2\theta}}{\cancel{a^2}}\,d\theta \qquad 1$$

$$= 2\pi = \oint_C = \oint_{C_a} \quad \#$$

Let $f = f(x,y)$ be a smooth function on $\mathbb{R}^2$

$$H = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{pmatrix}$$

Suppose λ is an eigenvalue of H,

i.e. ...

Def

Let $H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$, $\lambda \in \mathbb{R}$

Consider

$$(*_\lambda) \begin{cases} ax + by - \lambda x = (a-\lambda)x + by = 0 \\ cx + dy - \lambda y = cx + (d-\lambda)y = 0 \end{cases}$$

Note $\begin{cases} x = 0 \\ y = 0 \end{cases}$ is a sol. of $(*_\lambda)$

$\leftarrow$ zero sol

Q: Does $(*_\lambda)$ have another sol. ?
(nonzero sol.)

We say λ is an <u>eigenvalue</u> $\Leftrightarrow$

$(*_\lambda)$ has a nonzero sol.

That is, $\exists\ \vec{v} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ with $x_0 \neq 0$ or $y_0 \neq 0$

$\nwarrow$ called an eigenvector of H

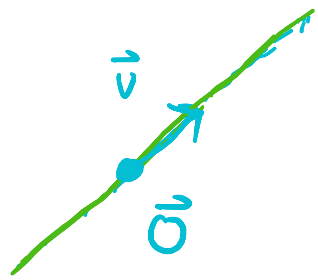
s.t.

$$H\vec{v} = \lambda \vec{v}$$

$$\Leftrightarrow \begin{cases} ax_0 + by_0 = \lambda x_0 \\ cx_0 + dy_0 = \lambda y_0 \end{cases}$$

Claim If f has a local min. at $(0,0)$ then $\lambda \geq 0$

Pf



Consider

$$g(t) := f(\vec{0} + t\vec{v}) = f(tx_0, ty_0)$$

Since f has a local min at $(0,0)$.

g has a local min. at $t=0$

$\Rightarrow g''(0) \geq 0$ (otherwise $g''(0) < 0$ $\overset{\text{2nd der. test}}{\Rightarrow}$ g has a local max at $t=0$

$\Rightarrow$ g is constant near $t=0$

$\Rightarrow g''(0) = 0$ $\rightarrow\leftarrow$)

Compute $g''(0)$:

$$g'(t) = \left(f(tx_0, ty_0)\right)'$$

$$\overset{\text{chain rule}}{=} \left(\nabla f(tx_0, ty_0)\right) \bullet (x_0, y_0)$$

$$= f_x(tx_0, ty_0) \cdot x_0 + f_y(tx_0, ty_0) \cdot y_0$$

$$\Rightarrow g''(0) = \frac{d}{dt} \left(\quad \right) \Big|_{t=0}$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \Big(\nabla (x_0 f_x)(tx_0, ty_0) \Big) \bullet (x_0, y_0)$$

$$+ \Big(\nabla (y_0 f_y)(tx_0, ty_0) \Big) \bullet (x_0, y_0) \Big|_{t=0}$$

$x_0 \cdot \lambda x_0$

$$= x_0 f_{xx}(0,0) \cdot x_0 + x_0 f_{xy}(0,0)\, y_0$$

$$+ y_0 f_{yx}(0,0)\, x_0 + y_0 f_{yy}(0,0)\, y_0$$

$y_0 \lambda y_0 =$

$$\Big(= (H \cdot \vec{v}) \bullet \vec{v} \Big)$$

assumption:

$H\vec{v} = \lambda \vec{v}$

$\Updownarrow$ $H\vec{v}$

$$\begin{cases} f_{xx}(0,0)\, x_0 + f_{xy}(0,0)\, y_0 = \lambda x_0 \\ f_{xy}(0,0)\, x_0 + f_{yy}(0,0)\, y_0 = \lambda y_0 \end{cases}$$

$(x_0, y_0) \neq (0,0) \Downarrow$

> 0

$$= \lambda (x_0^2 + y_0^2) = g''(0)$$

$$\Rightarrow \lambda = \frac{g''(0)}{x_0^2 + y_0^2} > 0$$

#

$$f_{xyx} = f_{xxy} = f_{yxx}$$

pf

$$f_{xyx} = (f_x)_{yx} \overset{\text{Thm}}{=} (f_x)_{xy} = f_{xxy}$$

$$\parallel$$

$$(f_{xy})_x \overset{\text{Thm}}{=} (f_{yx})_x = f_{yxx}$$

#