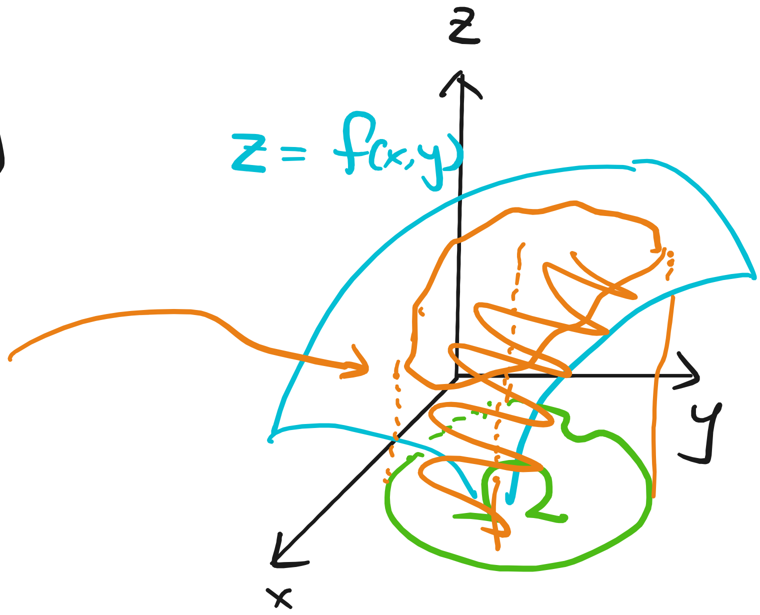
Calculus, Spring 2026, week 14

Recall

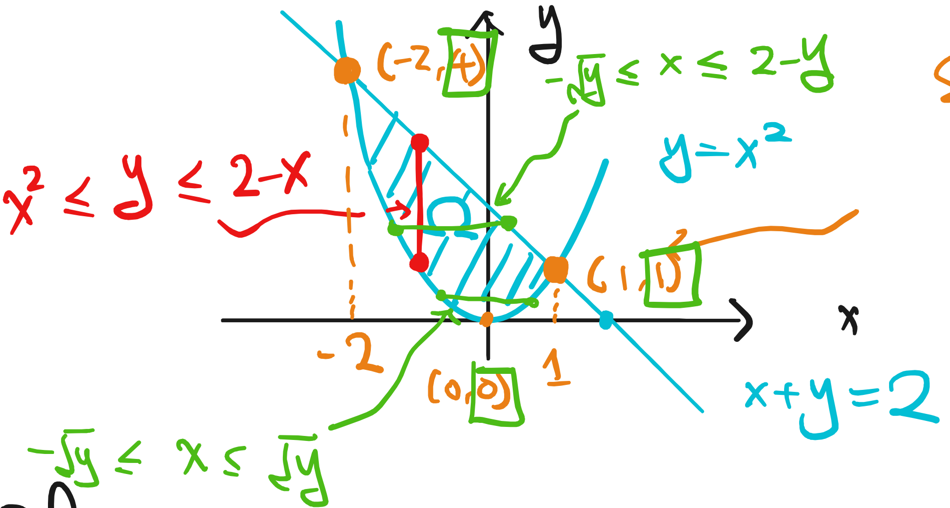
$$\iint_{\Omega} f(x,y)\,dx\,dy$$

= the volume of



Example

① Find the area of the region Ω



Solve

$$\begin{cases} y = x^2 \\ x + y = 2 \end{cases}$$

$\Rightarrow x + x^2 - 2 = 0$

$(x-1)(x+2) = 0$

$\Rightarrow x = -2, 1$

Sol

$$\text{area}(\Omega) = \iint_{\Omega} 1\,dx\,dy$$

$$= \int_{-2}^{1} \left(\int_{x^2}^{2-x} \underset{''}{1}\,dy \right) dx$$

y'

$$= \int_{-2}^{1} (2-x) - x^2 \; dx$$

$$= 2x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_{x=-2}^{1}$$

$$= \overset{6}{\cancel{2}} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \cancel{2(-2)} + \overset{\frac{3}{2}}{\cancel{\frac{4}{2}}} - \overset{-3}{\cancel{\frac{8}{3}}}$$

$$= \frac{9}{2}$$

#

Another method:

$$\text{area}(\Omega) = \iint_{\Omega} 1 \, dx \, dy$$

$$= \int_0^1 \left(\int_{-\sqrt{y}}^{\sqrt{y}} 1 \, dx \right) dy$$

$$+ \int_1^4 \left(\int_{-\sqrt{y}}^{2-y} 1 \, dx \right) dy$$

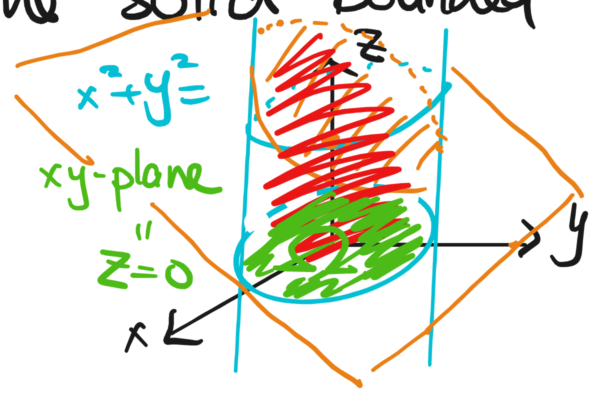
#

② Find the volume of the solid bounded by

$x^2+y^2=1$,

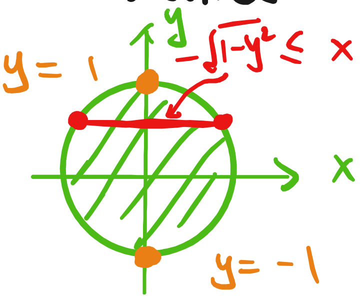
$\underline{y+z=2}$, $z=2-y$

$z=0$



Sol

$$\text{Volume} = \iint_{\Omega} 2-y \; dx\,dy$$



$$\Omega = \{x^2+y^2 \le 1\}$$

$$\Leftrightarrow \quad x^2 \le 1-y^2$$

$$\Leftrightarrow \quad -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$$

$$= \int_{-1}^{1}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2-y \; dx\right) dy$$

$$2-y = \frac{d}{dx}\left((2-y)x\right)$$

$$= \int_{-1}^{1}\left((2-y)x \,\Big|_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \right) dy$$

$$= \int_{-1}^{1} (2-y)\cdot 2\sqrt{1-y^2} \; dy$$

$$= 4\int_{-1}^{1}\sqrt{1-y^2}\, dy + \boxed{\int_{-1}^{1} \boxed{(-2)y\sqrt{1-y^2}}\, dy}$$

$1 = \cos 0$, $-1 = \cos\pi$

$y = \cos\theta \quad (\theta = \cos^{-1} y)$

$dy = -\sin\theta \, d\theta$

$$(-2)y\sqrt{1-y^2} = \left(\frac{2}{3}(1-y^2)^{\frac{3}{2}}\right)'$$

$$= \frac{2}{3}(1-y^2)^{\frac{3}{2}}\Big|_{-1}^{1} = 0$$

$$= 4\int_{\pi}^{0} \underbrace{\sqrt{1-\cos^2\theta}\,(-\sin\theta)}_{\substack{\parallel \\ -\sin^2\theta \,=\, -\frac{1-\cos 2\theta}{2}}} \, d\theta$$

$$= 4\int_{0}^{\pi} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= 4\cdot\left(\frac{\theta}{2} - \frac{1}{2}\,\frac{\sin 2\theta}{2}\right)\Bigg|_{0}^{\pi} = 4\left(\frac{\pi}{2} - 0\right)$$

$$= 2\pi \ \#$$

§ Triple integrals

Let $f: \mathbb{R}^3 \to \mathbb{R}$. Similarly to double integrals we can consider

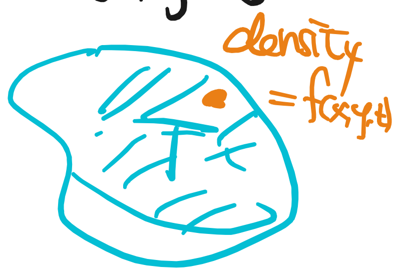
$$\iiint_T f(x,y,z)\, dx\,dy\,dz$$

can be defined by a limit of Riemann sums

where $T \subseteq \mathbb{R}^3$.

Geometrically, such an integral computes a "4-dimensional volume".

Alternatively, if one consider $f(x,y,z)$ a density function, then



$\iiint_T f(x,y,z)\,dxdydz$ computes the weight of T.

In particular, if $f(x,y,z) \equiv 1$, then

$$\iiint_T dxdydz = \iiint_T 1\, dxdydz$$

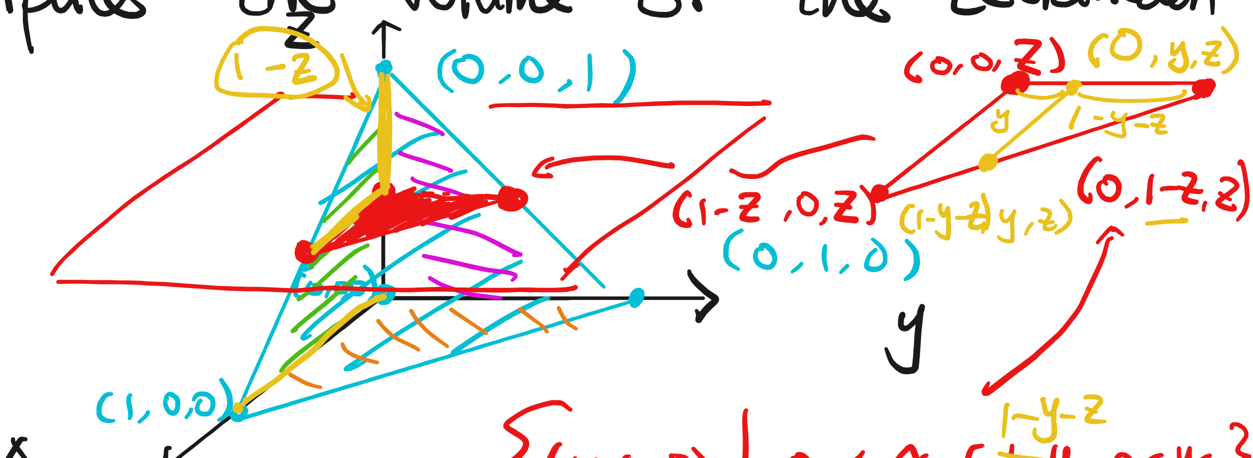
is the volume of T.

$\iiint_T f(x,y,z)\,dxdydz$
$\downarrow$

Similarly to double integrals, triple integrals can be repeated integrals.

Example

① Compute the volume of the tetrahedron T:



$\{(x,y,z) \mid 0 \le x \le 1-y-z,\ 0 \le y \le 1-z\}$

Sol

$$\text{volume} = \iiint_T 1 \, dx\,dy\,dz$$

Note that

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0 \le x \le \overset{\psi(y,z)}{\overset{\parallel}{\underline{1-y-z}}}, \\ 0 \le y \le \underline{1-z} = \phi(z), \\ 0 \le z \le 1 \end{array} \right\}$$

$$\Rightarrow \text{volume} = \int_0^1 \left(\int_0^{1-z} \underbrace{\left(\int_0^{1-y-z} 1 \, dx \right)}_{=1-y-z} dy \right) dz$$

$$= \int_0^1 \left((1-z)y - \frac{y^2}{2} \Big|_{y=0}^{1-z} \right) dz$$

$$= \int_0^1 (1-z)^2 - \frac{(1-z)^2}{2} \, dz$$

$$= -\frac{(1-z)^3}{6} \Big|_{z=0}^{1} = \frac{1}{6} \quad \#$$

② Integrate

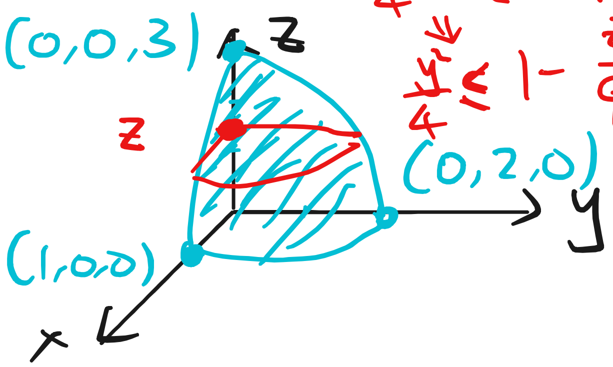
$$f(x,y,z) = xy$$

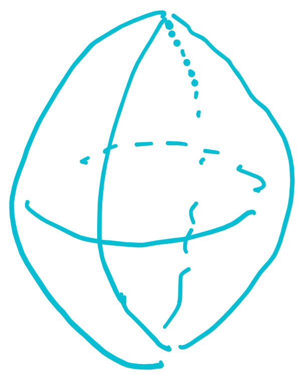
over the first-octant solid T bounded by the coordinate planes and

$$x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$$

$$\Rightarrow x^2+\frac{y^2}{4}=\left(1-\frac{z^2}{9}\right)$$

$$\Rightarrow \frac{y^2}{4}\le 1-\frac{z^2}{9}$$





Sol

$$T=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \begin{array}{l} 0\le z\le 3 \\ 0\le y\le \sqrt{4\left(1-\frac{z^2}{9}\right)} \\ 0\le x\le \sqrt{1-\frac{y^2}{4}-\frac{z^2}{9}} \end{array}\right\}$$

$$x^2=\left(1-\frac{y^2}{4}-\frac{z^2}{9}\right)$$

$$x\le\sqrt{1-\frac{y^2}{4}-\frac{z^2}{9}}$$

So

$$\text{ans}=\iiint_T xy \, dx\,dy\,dz$$

$$=\int_0^3\left(\int_0^{2\sqrt{1-\frac{z^2}{9}}}\left(\int_0^{\sqrt{1-\frac{y^2}{4}-\frac{z^2}{9}}} xy\,dx\right)dy\right)dz$$

$$=\cdots\text{ Computation }\cdots=\frac{4}{5}$$

#

Changing variables and Jacobians

Recall

$$\int_a^b f(\underbrace{u(x)}_{=u})\, u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

" $du = u'(x)\,dx$ "

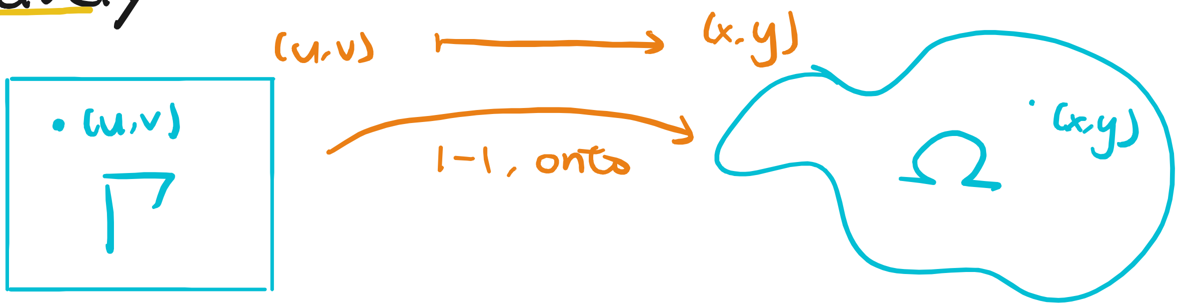
Thm (17.10.2)

Suppose

$$x = x(u,v)\ , \qquad y = y(u,v)$$

are smooth functions that map $\overset{(u,v)}{\Gamma}$ to $\overset{(x,y)}{\Omega}$ bijectively

1-1 and onto " bijectively



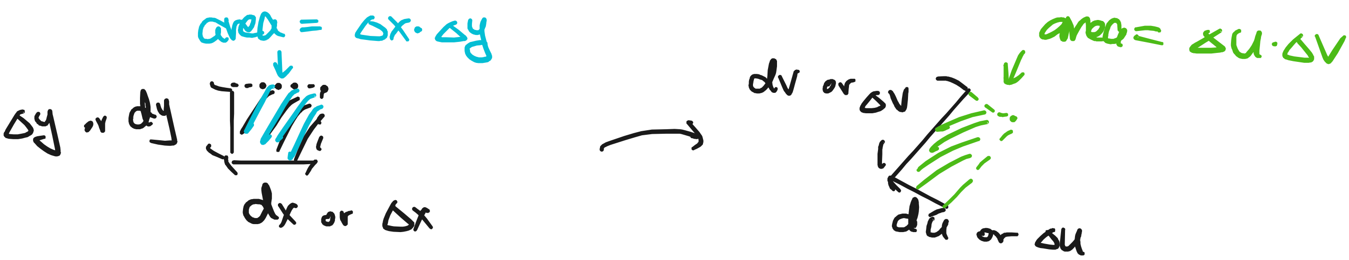
and the Jacobian

$$J(u,v) := \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial v}$$

is nonzero on Γ. Then

$$\iint_\Omega f(x,y)\,dx\,dy = \iint_\Gamma \underbrace{f(u,v)}_{=f(x(u,v),\,y(u,v))}\cdot |J(u,v)|\,du\,dv$$

Idea behind the formula:



Note that

$$\Delta x = x(u_i, v_i) - x(u_{i-1}, v_{i-1}) \overset{MVT}{=} \nabla x(c) \cdot (u_i - u_{i-1}, v_i - v_{i-1})$$

$$\text{where } x(u_i, v_i) = x_i, \quad x(u_{i-1}, v_{i-1}) = x_{i-1}$$

$$= \frac{\partial x(c)}{\partial u}\Delta u + \frac{\partial x(c)}{\partial v}\Delta v$$

$$\Delta y = \frac{\partial y(d)}{\partial u}\Delta u + \frac{\partial y(d)}{\partial v}\Delta v$$

$$\Rightarrow \quad \Delta x\,\Delta y \approx \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}\right| \cdot \Delta u \Delta v$$

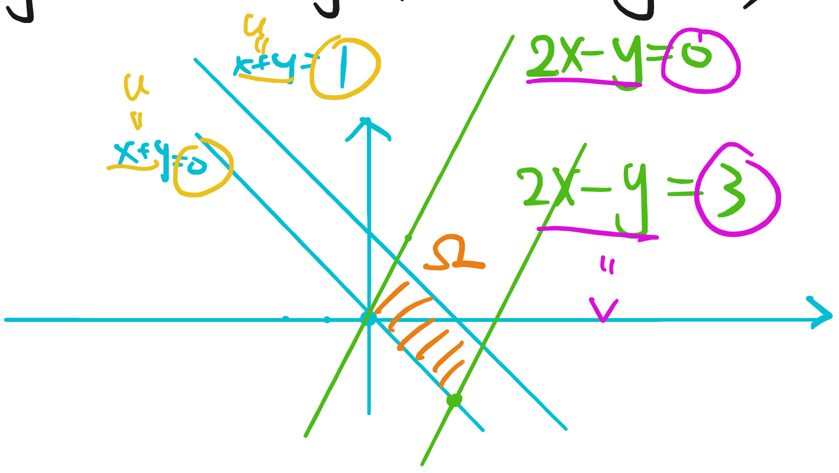
Example

$$\iint_\Omega (x+y)^2\,dx\,dy = ?$$

where Ω is the parallelogram bounded by

$x+y=0$, $x+y=1$, $2x-y=0$, $2x-y=3$

sol



Set

$$u = x+y \quad , \qquad v = 2x - y$$

$$\Leftrightarrow \quad x = \frac{u+v}{3} \quad , \qquad y = \frac{2u-v}{3}$$

This transformation defines a 1-1, onto map

$$\Gamma = \left\{ (u,v)\in\mathbb{R}^2 \;\middle|\; \begin{array}{l} 0\le u\le 1 \\ 0\le v\le 3 \end{array} \right\} \xrightarrow{1-1,\text{onto}} \Omega$$

The Jacobian is

$$J(u,v) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$= \frac{1}{3}\times\left(\frac{-1}{3}\right) - \frac{2}{3}\times\frac{1}{3} = \frac{-1}{9} - \frac{2}{9} = -\frac{1}{3}$$

$$\Rightarrow \iint_{\Omega} \overset{(x+y)^2}{\overset{\shortparallel}{f(x,y)}}\, dx\,dy = \iint_{\Gamma} \overset{u^2}{\overset{\shortparallel}{f(u,v)}} \left|-\frac{1}{3}\right| du\,dv$$

$$= \frac{1}{3}\iint_{\Gamma} u^2\, du\,dv = \frac{1}{3}\int_0^3 \Big(\underbrace{\int_0^1 u^2\, du}_{=\frac{1}{3}}\Big)\, dv$$

$$= \frac{1}{3}\times 3\times\frac{1}{3} = \frac{1}{3} \qquad \#$$

Let $\quad x = r\cos\theta, \quad y = r\sin\theta$

$$\Rightarrow \quad J(r,\theta) = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

$$= \det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

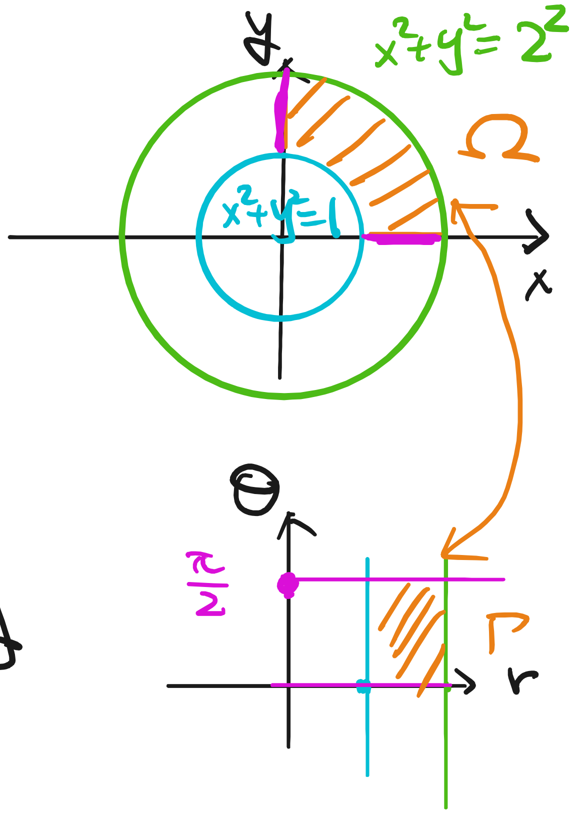
$$= r$$

if Ω and Γ are choosen properly

$$\Rightarrow \quad \iint_\Omega f(x,y)\,dx\,dy \overset{\downarrow}{=} \iint_\Gamma \underbrace{f(r,\theta)}_{\parallel} \, r\,dr\,d\theta$$

$f(r\cos\theta,\ r\sin\theta)$

<u>Example</u>



Let Ω be the region

$\text{Area}(\Omega) = ?$

<u>Sol</u>

$$\text{area}(\Omega) = \iint_\Omega 1\,dx\,dy$$

$$= \iint_\Gamma 1 \cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\boxed{\int_1^2 r\,dr} \right) d\theta \qquad = \frac{4}{2} - \frac{1^2}{2} = \frac{3}{2}$$

$$= \int_0^{\frac{\pi}{2}} \frac{3}{2}\,d\theta = \frac{3\pi}{4} \quad \#$$

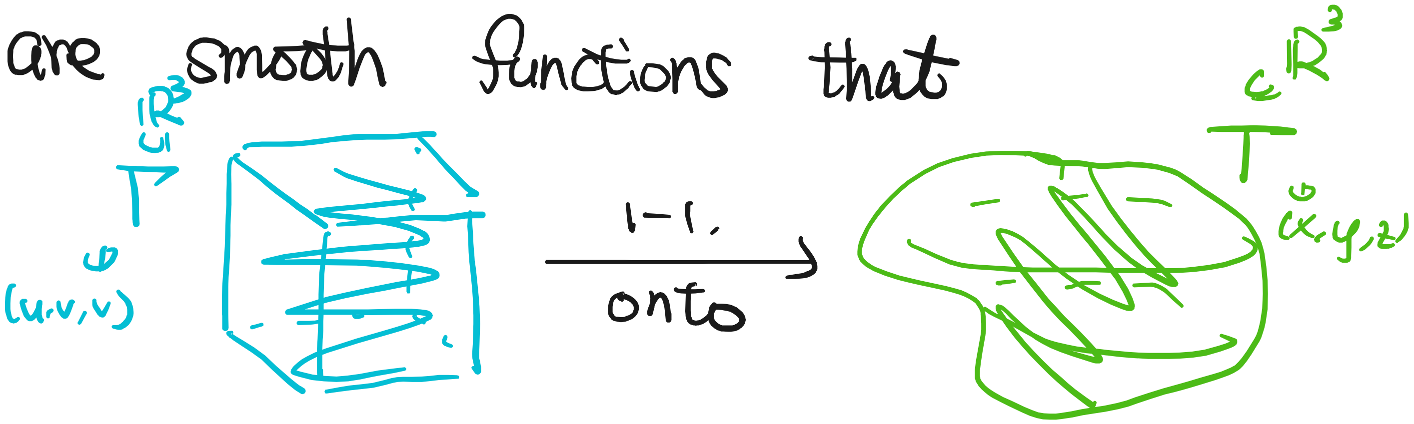
The case of 3 variables:

Thm (page 934)

Suppose

$$x = x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w)$$

are smooth functions that



and the Jacobian

$$J(u,v,w) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix}$$

$$= \frac{\partial x}{\partial v} \cdot \det\begin{pmatrix} \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix} - \frac{\partial x}{\partial v} \det\begin{pmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix}$$

$$+ \frac{\partial x}{\partial w} \cdot \det\begin{pmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$\neq 0 \quad \forall (u,v,w) \in \Gamma$

Then

$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_\Gamma \underbrace{f(u,v,w)}_{= f(x(u,v,w),\ y(u,v,w),\ z(u,v,w))} |J(u,v,w)|\,du\,dv\,dw$$

Example

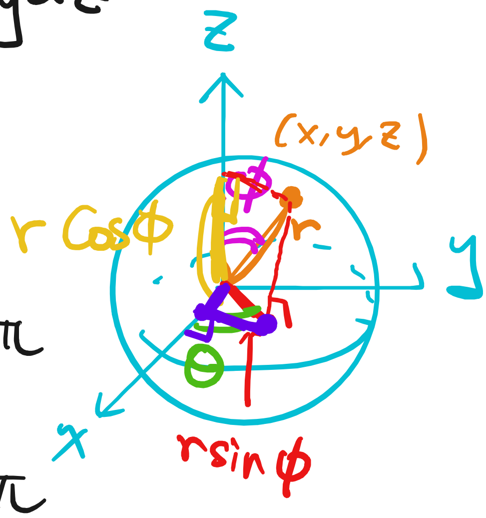
Let $B = \{ (x,y,z) \in \mathbb{R}^3 \mid \underbrace{x^2+y^2+z^2}_{= r^2} \le 1 \}$

$\text{Volume}(B) = ?$ $\left(= \frac{4}{3}\pi \right)$

sol

$$\text{Volume}(B) = \iiint_B 1\,dx\,dy\,dz$$

Consider the transformation



$$x = r\sin\phi\cos\theta \qquad 0 \le \theta < 2\pi$$

$$y = r\sin\phi\sin\theta \qquad 0 \le \phi \le \pi$$

$$z = r\cos\phi$$

"spherical coordinates"

The Jacobian is

$$J(r,\phi,\theta) = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{pmatrix}$$

$$= \det \begin{pmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \end{pmatrix}$$

$$= \sin\phi\cos\theta\left(r\cos\phi\sin\theta \cdot 0 - (-r\sin\phi)\, r\sin\phi\cos\theta\right)$$

$$- r\cos\phi\cos\theta\left(\sin\phi\cos\theta \cdot 0 - \cos\phi \cdot r\sin\phi\cos\theta\right)$$

$$+ (-r\sin\phi\sin\theta)\left(\sin\phi\sin\theta\,(-r\sin\phi) - \cos\phi\, r\cos\phi\sin\theta\right)$$

$$= \cdots = \underline{r^2\sin\phi}$$

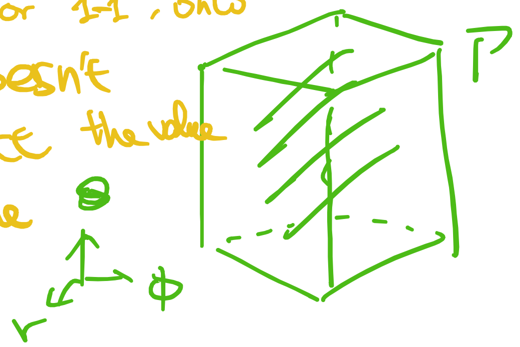
Since

$$B = \{\, \underbrace{x^2+y^2+z^2}_{= r^2} \le 1 \,\}_{(x,y)\neq(0,0)} = \left\{ \begin{array}{l} (x,y,z) \\ \quad = \\ (r\sin\phi\cos\theta, \\ \ r\sin\phi\sin\theta, \\ \ r\cos\phi) \end{array} \middle| \begin{array}{l} 0 < r \le 1 \\ 0 < \phi < \pi \\ 0 \le \theta < 2\pi \end{array} \right\}$$

z-axis

→ for 1-1, onto

it doesn't affect the value of volume



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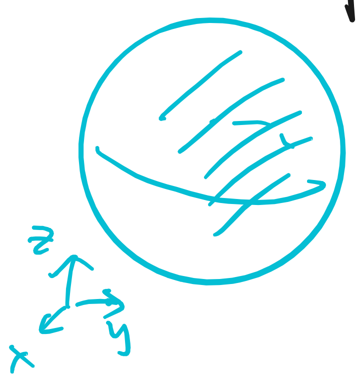

B

$$\Rightarrow \iiint_B dx\,dy\,dz = \iiint_\Gamma |J(r,\phi,\theta)|\, dr\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^1 \underbrace{r^2 \sin\phi}_{\frac{d}{dr}\left(\frac{r^3}{3}\sin\phi\right)} \, dr\right) d\phi\right) d\theta$$

$$= \frac{r^3}{3}\sin\phi \Big|_{r=0}^{1}$$

$$= \int_0^{2\pi}\left(\int_0^{\pi} \frac{1}{3} \overset{(-\cos\phi)'}{\boxed{\sin\phi}} \, d\phi\right) d\theta$$

$$= -\frac{\cos\phi}{3}\Big|_{\phi=0}^{\pi} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

$$= \int_0^{2\pi} \frac{2}{3} \, d\theta$$

$$= 2\pi \cdot \frac{2}{3} = \frac{4}{3}\pi \quad \#$$

Problem 1 Find the partial derivatives $\dfrac{\partial u}{\partial s}$ and $\dfrac{\partial u}{\partial t}$, where

$$u = x^2 - xy, \quad x = s\cos t, \quad y = t\sin s.$$

常錯 計算完成後記得把 x 跟 y 換成 s, t 的形式，不要留下以下形式：

$$\frac{\partial u}{\partial s} = (2x - y)\cos t - xt\cos s.$$

Average	48.92920354
Average (except zero)	53.16346154
Quartile 0	0
Quartile 1	49
Quartile 2	54
Quartile 3	59
Quartile 4	60
標準差	15.9013234
非零數	104
不到50%人數	10
滿分人數	22