Calculus, Spring 2026, week 13

Thm (Thm 16.5.2)

If f has a local extreme value at $\vec{v}_0$, then

$$\nabla f(\vec{v}_0) = \vec{0}$$

or

$$\nabla f(\vec{v}_0) \text{ does not exist}$$

pf

Assume $\nabla f(\vec{v}_0) = (f_{x_1}(\vec{v}_0), \dots, f_{x_n}(\vec{v}_0))$ exists. We need to show $\nabla f(\vec{v}_0) = \vec{0}$.

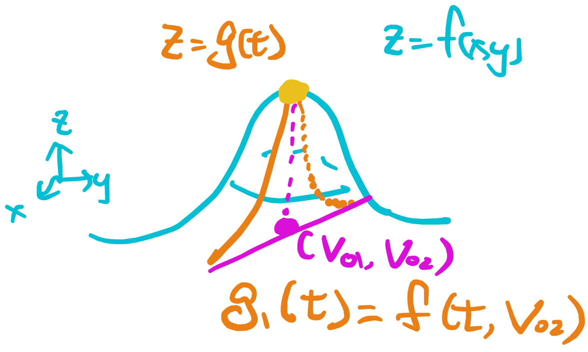
Let $\vec{v}_0 = (v_{01}, v_{02}, \dots, v_{0n}) \in \mathbb{R}^n$

Consider

i-th component

$$g_i(t) := f(v_{01}, \dots, v_{0(i-1)}, t, v_{0(i+1)}, \dots, v_n)$$

$i = 1, \dots, n.$



Then g_i has a local extreme value at $t = v_{0i}$

$$\Rightarrow \quad g_i'(v_{0i}) = f_{x_i}(\vec{v}_0) = 0, \quad i = 1, \dots, n$$

using the result one-variable functions :

$g(t)$ has a local extreme value at t_0

$\Rightarrow g'(t_0) = 0$ or $g'(t_0)$ doesn't exist.

$\Rightarrow \nabla f(\vec{v}_0) = \vec{0}$ #

Def

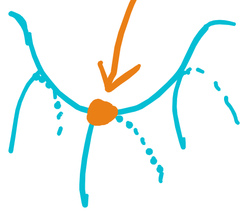
A point $\vec{v}_0$ is ① called a critical point if $\nabla f(\vec{v}_0) = \vec{0}$ or $\nabla f(\vec{v}_0)$ doesn't exist

② is called a stationary point if $\nabla f(\vec{v}_0) = \vec{0}$

③ is called a saddle point 馬鞍點 if $\nabla f(\vec{v}_0) = \vec{0}$ and f does NOT have a local extreme value at $\vec{v}_0$.

e.g.

$z=f(x,y)$



Example

$$f(x,y) = 2x^2 + y^2 - xy - 7y$$

has the gradient

$$\nabla f(x,y) = (4x - y, \; 2y - x - 7)$$

Then $\nabla f(x,y) = (0,0)$ $\Leftrightarrow$ $\begin{cases} 4x - y = 0 \\ 2y - x - 7 = 0 \end{cases}$

$$\Leftrightarrow \begin{cases} y = 4x \\ 8x - x - 7 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 4 \end{cases}$$

So $(1,4)$ is the only critical point of $f(x,y)$.

Q: Does f have a (local) extreme value at $(1,4)$?

Note that

$$f(1+h, 4+k) - f(1,4)$$

$$= 2(1+h)^2 + (4+k)^2 - (1+h)(4+k) - 7(4+k)$$

$$- (2 \cdot 1^2 + 4^2 - 1 \cdot 4 - 7 \cdot 4)$$

$$= \cancel{4h} + 2h^2 + \cancel{8k} + k^2 - hk \cancel{-k} - \cancel{4h} \cancel{-7k}$$

$$= 2h^2 + k^2 - hk = h^2 + h^2 \underline{- hk} + k^2$$

$$\geq h^2 + h^2 \underline{- 2|h| \cdot |k|} + k^2$$

$$= h^2 + (|h| - |k|)^2 \geq 0 \quad \forall h, k \in \mathbb{R}$$

So f has a minimum $f(1,4) = -14$ at $(1,4)$

\#

__Recall__

If $g \overset{\in C^2}{=} g(x)$ and $g'(x_0) = 0$, then

(i) $g''(x_0) > 0 \Rightarrow g$ has a local minimum at x_0

$y = g(x)$ $\cup$

(ii) $g''(x_0) < 0 \Rightarrow g$ has a local maximum at x_0

$\cap$

__Thm__ (Thm 16.5.3)

Suppose $f = f(x,y)$ is smooth (or C^2). Set

$$A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$$

$$B = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

$$C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

and

$$D = AC - B^2$$

Then

(i) If $D < 0$, then (x_0, y_0) is a saddle point.

(ii) If $D > 0$, then

$\Rightarrow A \neq 0$

(a) $A>0 \Rightarrow$ f has a local minimum at (x_0, y_0)

(b) $A<0 \Rightarrow$ f has a local maximum at (x_0, y_0)

In the previous example,

$f_x = 4x - y \Rightarrow f_{xy} = -1$

$$f(x,y) = 2x^2 + y^2 - xy - 7y$$

$$\nabla f(1,4) = \vec{0}$$

$$A = f_{xx}(1,4) = \underline{4}^{\,>0}, \quad B = f_{xy}(1,4) = -1$$

$$C = f_{yy}(1,4) = 2$$

$$\Rightarrow D = AC - B^2 = 4 \times 2 - (-1)^2 = \underline{7 > 0}$$

$\Rightarrow$ f has a local minimum at $(1,4)$. #

Remark

$$D = AC - B^2 = f_{xx} f_{yy} - (f_{xy})^2$$

$$= \det \underbrace{\begin{pmatrix} f_{xx}^{\,=A} & f_{xy}^{\,=B} \\ f_{yx}^{\,=B} = f_{xy} & f_{yy}^{\,=C} \end{pmatrix}}_{\text{second derivative of } f} = E$$

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

symmetric matrix

In general, 2^{nd} derivative of $f = f(x_1, \dots, x_n)$

$$= E = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & & & \vdots \\ \vdots & & & \\ f_{x_nx_1} & \cdots & & f_{x_nx_n} \end{pmatrix}_{n\times n}$$

Since $E = E^T$, it is diagonalizable
(o.g.)
i.e. $\exists$ invertible matrix P and $\lambda_1, \lambda_2 \in \mathbb{R}$
s.t.

$$E = P^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P$$

n variable case:
$$E = P^{-1} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P$$

The conditions in Thm are:

不定 eigenvalues 有正有負

(i) $D < 0 \iff \underbrace{\lambda_1 \lambda_2}_{} < 0$

$= \det(E) = \cancel{\det(P^{-1})} \det\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \cancel{\det(P)} = \lambda_1 \lambda_2$

Symmetric matrix

正定 (ii)-(a) $AC - B^2 = D > 0$, $\Rightarrow A > 0 \iff \lambda_1, \lambda_2 > 0$ ($D = \lambda_1\lambda_2$)

" eigenvalues 全正

$AC > B^2 \geq 0 \Rightarrow AC > 0 \Rightarrow A + C > 0 \iff A > 0$

$A + C < 0 \iff A < 0$

(ii)-(b) $D > 0$, $A < 0 \iff \lambda_1, \lambda_2 < 0$ ($D = \lambda_1\lambda_2$)

負定 = eigenvalues 全負

Recall λ_1, λ_2 are sol. of

$$\det(E - \lambda I) = \det\begin{pmatrix} A - \lambda & B \\ B & C - \lambda \end{pmatrix}$$

$$= \lambda^2 - \underbrace{(A+C)}_{\lambda_1 + \lambda_2} \lambda + \underbrace{(AC - B^2)}_{= \lambda_1 \lambda_2}$$

Remark

$g(t) = f(\vec{v}_0 + t\vec{w})$, $g'(t) = \nabla f(\vec{v}_0 + t\vec{w}) \cdot \vec{w}$

$= f_x(v_1 + tw_1, v_2 + tw_2) \cdot w_1 + f_y(\cdots) \cdot w_2$

$$g''(t) = f_{xx} \cdot w_1 w_2 + f_{xy}\, w_1 w_2 + f_{yx}\, w_2 w_1 + f_{yy}\, w_2 w_2$$

E

$$g''(0) = (w_1 \;\; w_2) \underline{\begin{pmatrix} f_{xx}(\vec{v}_0) & f_{xy}(\vec{v}_0) \\ f_{yx}(\vec{v}_0) & f_{yy}(\vec{v}_0) \end{pmatrix}} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

If $E\vec{w} = \lambda \vec{w}$, then $g''(0) = (w_1 \;\; w_2) \begin{pmatrix} \lambda w_1 \\ \lambda w_2 \end{pmatrix}$

$$= \lambda (w_1^2 + w_2^2) = \lambda \underbrace{\|\vec{w}\|^2}_{>0}$$

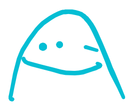
Example

① $f(x,y) = x^2 + y^2$

$\nabla f = (2x, 2y) = (0,0) \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$E = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ local min. at $(0,0)$

② $f(x,y) = -x^2 - y^2$



$\nabla f = (-2x, -2y) = \vec{0} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$E = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$ local max at $(0,0)$

③ $f(x,y) = xy$

$\nabla f = (y, x) = \vec{0} \Leftrightarrow (x,y) = (0,0)$

$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow D < 0 \Rightarrow$ saddle point at $(0,0)$.

Global / absolute extreme values

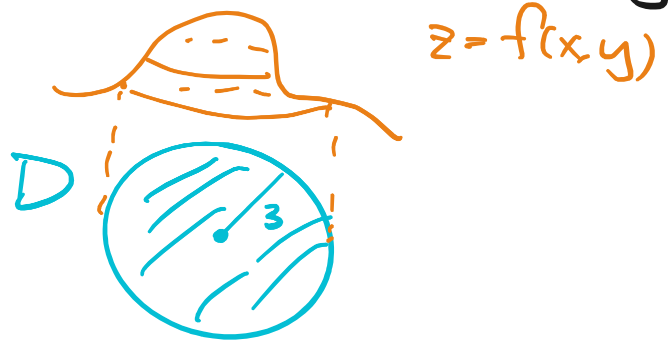
Example

Find the absolute extreme values taken by the function

$$f(x,y) = x^2 + y^2 - 2x - 2y + 4$$

on the closed disk

$$D = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9 \}$$



Sol

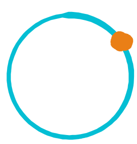
Step 1: Find critical points (local extreme values) in the interior $\mathrm{int}(D) = \{ x^2 + y^2 < 9 \}$

$$Df = (2x - 2,\ 2y - 2) = (0,0)$$

$$\Leftrightarrow (x,y) = (1,1) \in \mathrm{int}(D)$$

Step 2: Find extreme values on the boundary

$$x^2 + y^2 = 9$$



$\vec{r}(t) = (3\cos t,\ 3\sin t)$

So we shall find the extreme values of

$$F(t) = f(\vec{r}(t)) = f(3\cos t,\ 3\sin t)$$
$$= 13 - 6\cos t - 6\sin t$$
$$F'(t) = 6\sin t - 6\cos t = 0 \iff t = \frac{\pi}{4} + n\pi$$

$\Rightarrow$ The extreme values of f occur at

$$\vec{r}\left(\frac{\pi}{4} + n\pi\right) = \left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right) \text{ or } \left(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$$

<u>Step3</u> Compare the values of f at all these points:

$$f(1,1) = 2 \leftarrow \text{global min.}$$
$$f\left(\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2}\right) = 13 - 6\sqrt{2} \approx 4.5$$
$$f\left(-\frac{3}{2}\sqrt{2}, -\frac{3}{2}\sqrt{2}\right) = 13 + 6\sqrt{2} \approx 21.5$$

global max.

\#

<u>Remark</u>

Consider $f(x,y) = x$ on $(0,1) \times (0,1)$

In <u>Step3</u>, we implicitly assumed that f attains both a max. and min. value in $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$ ← $\|\vec{v}\| \leq 3$ $\forall\ \vec{v} \in D$

In fact, this assumption is true

because the following theorem:

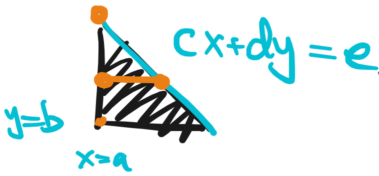
If f is a continuous function on a "closed" and "bounded" subset K of $\mathbb{R}^n$ (such as a closed disk

$$\{(x-a)^2+(y-b)^2 \le r^2\}$$

or a closed rectangle

$$[a,b]\times[c,d] = \left\{ (x,y)\in\mathbb{R}^2 \;\middle|\; \begin{array}{l} a\le x\le b \\ c\le y\le d \end{array} \right\}$$

or a closed triangle



$$\left\{ (x,y)\in\mathbb{R}^2 \;\middle|\; \begin{array}{l} a\le x\le \frac{e-dy}{c} \\ b\le y\le \frac{e-ca}{d} \end{array} \right\} \Bigg)$$

then f attains both a max. and a min. value on K.

Lagrange multiplier

In the previous example, in Step 2, we found extreme values of f on

$$\{x^2+y^2=9\} = C$$

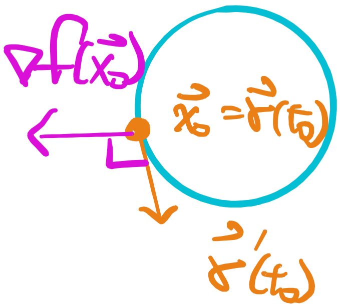
Note that if $\vec{\gamma}:\mathbb{R}\to C$ is a smooth parametrization of C (e.g. $\vec{\gamma}(t)=(3\cos t, 3\sin t)$) and if $\vec{x}_0=\vec{\gamma}(t_0)$ maximizes or minimizes f

then

$$\left.\frac{d}{dt}\right|_{t=t_0} f(\vec{\gamma}(t)) = 0$$

chain rule

$$\left.\frac{d}{dt}\right|_{t=t_0} f(\vec{\gamma}(t)) = \nabla f(\vec{x}_0) \cdot \vec{\gamma}\,'(t_0) = 0$$



So to find $\vec{x}_0$, it suffices to find the points at which ∇f is normal to C.

So if C is given by

$$g(x,y) = d$$

then $\nabla g \parallel \nabla f$. (or $\lambda \nabla g = \nabla f$ for some $\lambda \in \mathbb{R}$)

Lagrange multiplier

(assume $\nabla g \neq \vec{0}$)

Example

Consider

$$f(x,y) = x^2 + y^2 - 2x - 2y + 4$$

on $C = \{ \overbrace{x^2+y^2}^{g(x,y)} = 9 \}$

(subject to the constraint $x^2+y^2=9$)

<u>Sol</u> (Extreme values of f)

By the above discussion, we want to find

$$(x,y) \in C$$

s.t.

$$\nabla f(x,y) \quad // \quad \nabla g(x,y)$$

where $g(x,y) = x^2 + y^2$

$$\nabla f = (2x-2, \; 2y-2)$$

$$\nabla g = (2x, \; 2y)$$

$$\nabla f \; // \; \nabla g \iff 2x(2y-2) = (2x-2)2y$$

$$\Rightarrow x = y$$

On $C = \{x^2+y^2=9\}$, $\exists$ 2 points satisfying $x=y$:

$$2x^2 = x^2+y^2 = 9 \Rightarrow x = \pm\sqrt{\frac{9}{2}}$$

$$\Rightarrow \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), \quad \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$$

#

<u>Example</u>

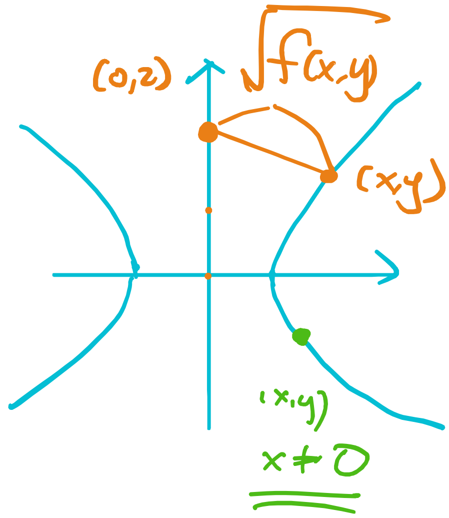
Find the minimum value of

$$f(x,y) = x^2 + (y-2)^2 \leftarrow$$

with the constrain $x^2-y^2=1$

$\Updownarrow$

$g(x,y)=0$



sol

Set

$$g(x,y)=x^2-y^2-1$$

Solve (x,y) with the property

$$\nabla f \parallel \nabla g$$

$$\nabla f = (2x,\ 2y-4), \qquad \nabla g = (2x,\ -2y) \neq (0,0) \text{ if } g(x,y)=0$$

$\Rightarrow \exists\, \lambda \in \mathbb{R}$ s.t.

$$\nabla f = \lambda \nabla g$$

$$(2x,\ 2y-4) = (\lambda 2x,\ -\lambda 2y)$$

Solve

$$\begin{cases} 2x = \lambda 2x \Rightarrow \lambda = 1 \\ 2y-4 = -\lambda 2y \Rightarrow 2y-4=-2y \Rightarrow y=1 \end{cases}$$

Use $g(x,y)=0=x^2-y^2-1$

$$\Rightarrow x = \pm\sqrt{2}$$

So the minimum value of f may happen at $(\sqrt{2}, 1)$ or $(-\sqrt{2}, 1)$

Check the value of f:

$$f(\sqrt{2}, 1) = 3$$

$$f(-\sqrt{2}, 1) = 3$$

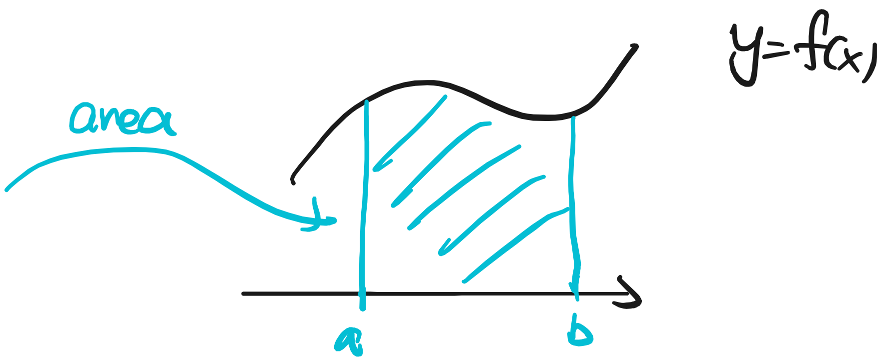
So the minimum value of f is 3 #

(二) 重積分

Double integrals

Recall

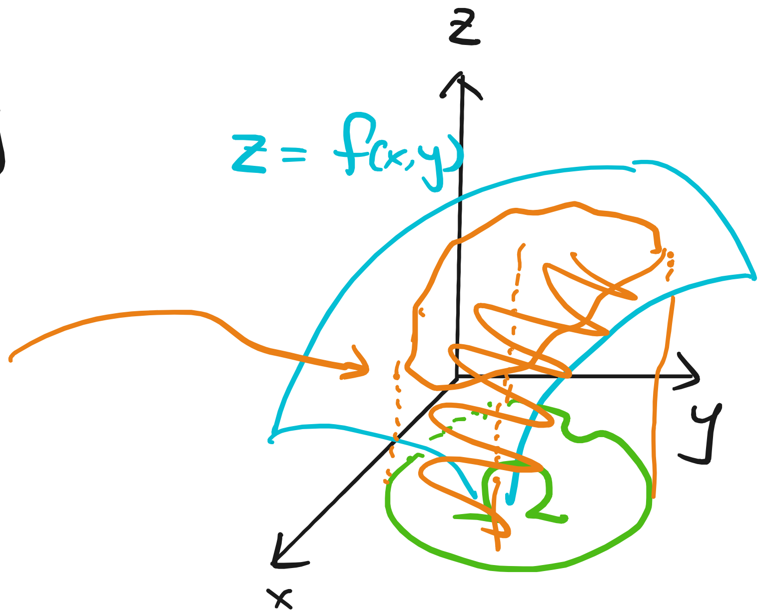
$$\int_a^b f(x)\,dx$$



Parallelly,

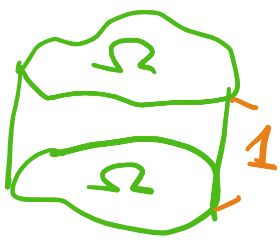
$$\iint_\Omega f(x,y)\,dx\,dy$$

= the volume of



In particular,

$$\iint_\Omega 1\,dx\,dy = \text{area}(\Omega) \times 1$$



An integral of this type is called a double integral, and it can be defined by a limit of "two-dimensional Riemann sum".

The precise definition is complicated and we skip it here.

In practice, double integrals are computed by "repeated integrals":

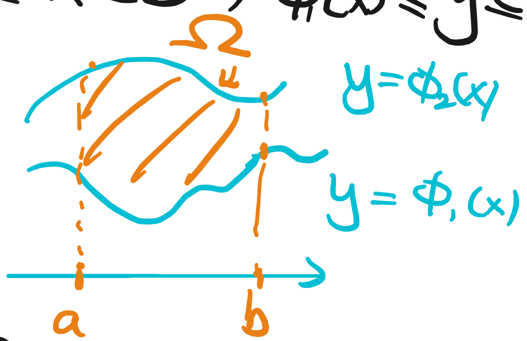
Thm (Cor 9.22 in Marsden's book)

Let $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ be continuous maps s.t. $\phi_1(x) \leq \phi_2(x)$ $\forall x \in [a,b]$.

Assume

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, \ \phi_1(x) \leq y \leq \phi_2(x)\}$$

and $f : \Omega \to \mathbb{R}$ is continuous.



Then

$$\iint_\Omega f(x,y)\,dx\,dy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$

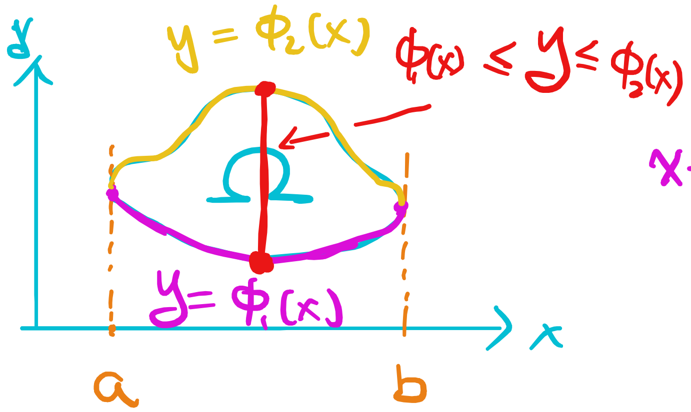
Also see (17.3.1) and (17.3.2) in textbook

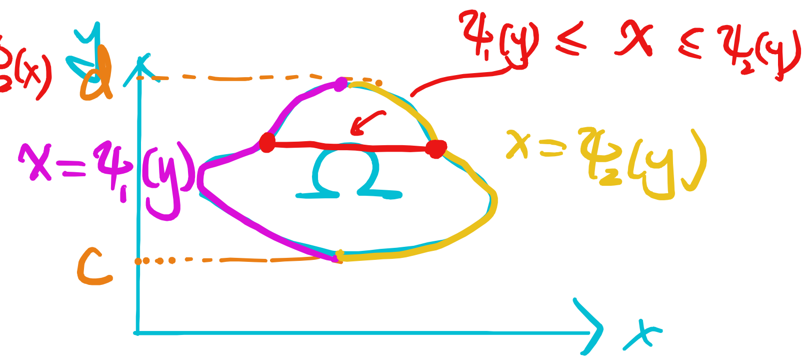
Remark

$\exists$ 2 ways to Compute $\iint_\Omega f(x,y)\,dx\,dy$:

① $$\iint_\Omega f(x,y)\,dx\,dy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$

② $$\iint_\Omega f(x,y)\,dx\,dy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy$$





Example

① $\iint_R (x+y-2)\,dx\,dy$, $\quad R = \left\{ \begin{array}{l} 1 \le x \le 4 \\ 1 \le y \le 3 \end{array} \right\}$

$$\overset{\text{Rmk②}}{=} \int_1^3 \left(\int_1^4 (x+y-2)\,dx \right) dy$$

$(x+y-2) = \frac{d}{dx}\left(\frac{x^2}{2} + (y-2)x\right)$

$$= \int_1^3 \left. \frac{x^2}{2} + (y-2)x \right|_{x=1}^{4} dy$$

$$= \int_1^3 \underbrace{\cancel{\frac{16}{2}}^{\,8} + \cancel{(y-2)4} - \frac{1}{2} - (y-2)\cdot 1}_{3y + \frac{3}{2}}\, dy$$

$$= 3\frac{y^2}{2} + \frac{3y}{2}\Big|_{y=1}^{3} = 3\cdot\frac{9}{2} + \frac{3}{2}3 - \cancel{3\frac{1}{2} - \frac{3}{2}1}$$

$$= 15 \quad \#$$

Another method (Rmk ①)

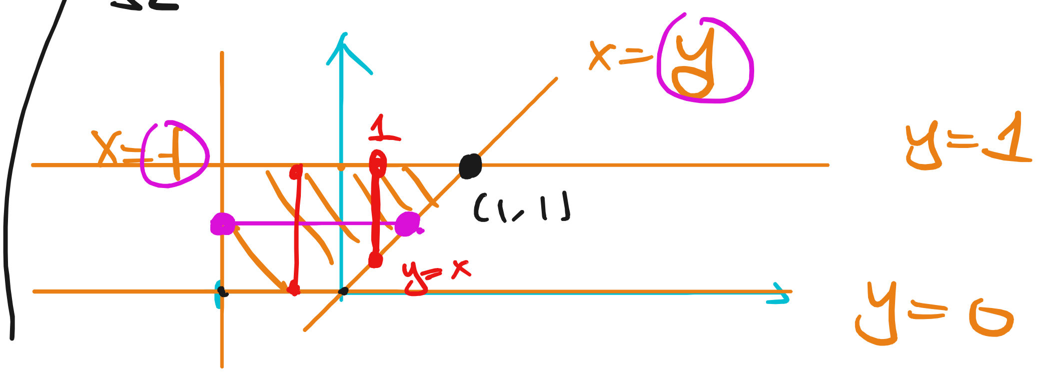
$$= \int_1^4 \left(\int_1^3 \underbrace{(x+y-2)}_{\frac{d}{dy}\left(\frac{y^2}{2} + (x-2)y\right)} dy \right) dx$$

$$= \int_1^4 \frac{y^2}{2} + (x-2)y \Big|_{y=1}^{3} dx \qquad = \frac{9}{2} + (x-2)3 - \frac{1}{2} - (x-2)\cdot 1$$

$$= \int_1^4 \underbrace{\cancel{4 + 2(x-2)}}_{}^{\,2x}\, dx = x^2\Big|_{x=1}^{4} = 16 - 1$$

$$= 15 \quad \#$$

② $\iint_\Omega (xy - y^3)\, dx dy$, $\Omega =$ the region enclosed by $y=0$, $y=1$, $x=-1$, $x=y$



$$\overset{\text{Rmk②}}{=} \int_0^1 \left(\int_{-1}^{y} \underbrace{(xy - y^3)}_{\frac{d}{dx}\left(\frac{x^2}{2}y - y^3 x\right)} \, dx \right) dy$$

$$= \frac{x^2}{2}y - xy^3 \Big|_{x=-1}^{y} = \frac{y^3}{2} - y^4 - \frac{y}{2} - y^3$$

$$= \int_0^1 -y^4 - \frac{y^3}{2} - \frac{y}{2} \, dy = -\frac{y^5}{5} - \frac{y^4}{8} - \frac{y^2}{4} \Big|_{y=0}^{1}$$

$$= -\frac{1}{5} - \frac{1}{8} - \frac{1}{4} = -\frac{8+5+10}{40} = -\frac{23}{40} \ \#$$

$$\overset{\text{Rmk①}}{=} \int_{-1}^{0} \left(\int_0^1 (xy - y^3) \, dy \right) dx$$

$$+ \int_0^1 \left(\int_x^1 (xy - y^3) \, dy \right) dx = \cdots$$