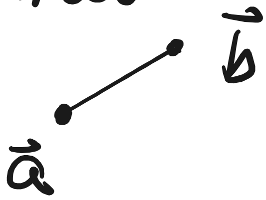
Calculus, Spring 2026, week 12

§ MVT and chain rule

Thm (MVT, Thm 16.3.1)

Let f be a smooth function on $\mathbb{R}^n$.

Then $\exists\ \vec{c} \in$



$= \{\vec{a} + t(\vec{b}-\vec{a}) \mid t\in[0,1]\} \subseteq \mathbb{R}^n$

s.t.

$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b}-\vec{a})$$

Recall (MVT) $f \in C^1[a,b]$

$\exists\ c$ between a and b s.t.

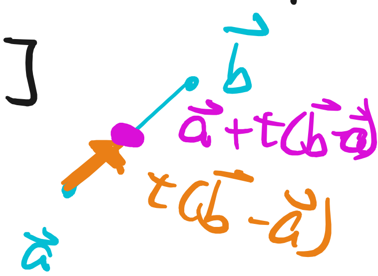
$$f(b) - f(a) = f'(c)\cdot(b-a)$$

Pf

If $\vec{a} = \vec{b}$, then LHS $= 0 =$ RHS.

Assume $\vec{a} \neq \vec{b}$. Recall that the line segment between $\vec{a}$ and $\vec{b}$ can be parametrized by

$$\vec{a} + t\,\underline{(\vec{b}-\vec{a})}, \quad t\in[0,1]$$



Let

$$\vec{u} = \frac{\vec{b}-\vec{a}}{\|\vec{b}-\vec{a}\|}$$

and

$$g(t) := f(\underbrace{\vec{a} + t(\vec{b}-\vec{a})}_{\vec{\gamma}(t)}) \ , \ t \in [0,1]$$

$$= f(\vec{a} + t\|\vec{b}-\vec{a}\| \ \vec{u}) = h(\|\vec{b}-\vec{a}\| t)$$

Last week:

The function $\underbrace{f(\vec{a}+t\vec{u})}_{= h(t)}$ is smooth and

$$\frac{d}{dt}\left(f(\vec{a}+t\vec{u})\right) = \nabla f(\vec{a}+t\vec{u}) \bullet \vec{u}$$

$\Rightarrow$ $g(t) = h(\|\vec{b}-\vec{a}\| \cdot t)$ is also smooth and

$$g'(t) \overset{\text{chain rule}}{=} h'(\|\vec{b}-\vec{a}\| \cdot t) \cdot \|\vec{b}-\vec{a}\|$$

$$= \left(\nabla f(\vec{a} + \underbrace{(\|\vec{b}-\vec{a}\| \cdot t) \cdot \vec{u}}_{\vec{b}-\vec{a}}) \bullet \underbrace{\vec{u}\right) \|\vec{b}-\vec{a}\|}_{(\vec{b}-\vec{a})}$$

$$= \nabla f(\vec{a} + t(\vec{b}-\vec{a})) \bullet (\vec{b}-\vec{a}).$$

Apply MVT to $g(t)$ (on $[0,1]$):

$\exists \ t_0 \in (0,1)$ s.t.

$$g(1) - g(0) = g'(t_0) \cdot (1-0)$$

$$\| \qquad\qquad\qquad \|$$

$$f(\vec{b}) - f(\vec{a}) \qquad \nabla f(\vec{a} + t_0(\vec{b}-\vec{a})) \bullet (\vec{b}-\vec{a})$$

Take $\vec{c} = \vec{a} + t_0(\vec{b}-\vec{a}) \Rightarrow \nabla f(\vec{c}) \bullet (\vec{b}-\vec{a})$ #

Cor

If $\nabla f(\vec{x}) = \vec{0}$ $\forall \vec{x} \in \mathbb{R}^n$, then

f is smooth

f is constant.

pf

$\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, $\exists \vec{c} \in \overline{xy}$ s.t.

$$f(\vec{x}) - f(\vec{y}) = \underbrace{\nabla f(\vec{c})}_{=0} \bullet (\vec{x} - \vec{y})$$

$$= 0$$

$\Rightarrow f(\vec{x}) = f(\vec{y})$ $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$ #

Cor

$\Rightarrow \nabla(f-g)(\vec{x}) = 0$ $\forall \vec{x} \in \mathbb{R} \Rightarrow f-g = \text{const}$

If $\nabla f(\vec{x}) = \nabla g(\vec{x})$ $\forall \vec{x} \in \mathbb{R}^n$, then $\exists c \in \mathbb{R}$

s.t. $f(\vec{x}) - g(\vec{x}) = c$ $\forall \vec{x} \in \mathbb{R}^n$

Thm (Chain Rule, Thm 16.3.4)

Let $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be smooth functions. Then

$$f \circ \vec{\gamma} : \mathbb{R} \to \mathbb{R}$$

is also smooth, and

if $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$

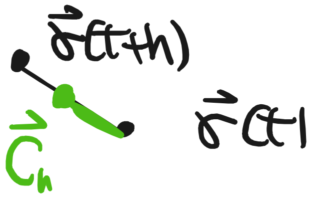
$$(f \circ \vec{\gamma})'(t) = \left(\nabla f(\vec{\gamma}(t))\right) \bullet \vec{\gamma}'(t)$$

$$= f_{x_1}(\vec{\gamma}(t))\,\gamma_1'(t) + f_{x_2}(\vec{\gamma}(t))\,\gamma_2'(t) + \dots + f_{x_n}(\vec{\gamma}(t))\,\gamma_n'(t).$$

Pf

Recall

$$(f\circ\vec{r})'(t) = \lim_{h\to 0} \frac{f(\vec{r}(t+h)) - f(\vec{r}(t))}{h}$$

By MVT, $\exists\ \vec{c}_h \in$ 

s.t.

$$\frac{f(\vec{r}(t+h)) - f(\vec{r})}{h} = \nabla f(\vec{c}_h) \bullet \left(\frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right)$$

$\to \vec{r}\,'(t)$ as $h \to 0$

?

Since

$\because \vec{r}$ is continuous (smooth $\Downarrow$)

$$0 \le \| \vec{c}_h - \vec{r}(t) \| \le \| \vec{r}(t+h) - \vec{r}(t) \| \to 0 \text{ as } h \to 0$$

we have

$$\lim_{h\to 0} \| \vec{c}_h - \vec{r}(t) \| = 0$$

i.e.

$$\lim_{h\to 0} \vec{c}_h = \vec{r}(t)$$

$$\Rightarrow \lim_{h\to 0} \nabla f(\vec{c}_h) = \nabla f(\vec{r}(t))$$

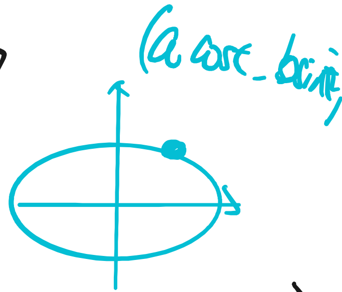
$\because f$ is smooth $\Rightarrow$ all f_{x_i} are continuous

$$\Rightarrow (f\circ\vec{r})'(t) = \lim_{h\to 0} \frac{f(\vec{r}(t+h)) - f(\vec{r}(t))}{h} = \nabla f(\vec{r}(t)) \bullet \vec{r}\,'(t) \quad \#$$

Example

① Find the rate of change of

$$f(x,y) = \frac{1}{3}(x^3+y^3)$$



with respect to t along $\vec{r}(t) = (a\cos t, b\sin t)$

sol

$$\text{Ans} = (f\circ r)'(t) = \big(f(a\cos t, b\sin t)\big)'$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \big(\nabla f(a\cos t, b\sin t)\big) \bullet (-a\sin t,\ b\cos t)$$

$$= f_x(a\cos t, b\sin t)(-a\sin t)$$

$$+ f_y(a\cos t, b\sin t)\ b\cos t$$

$$= \cdots = \sin t\cos t\,(b^3\sin t - a^3\cos t) \qquad \#$$

② Let $u = x^2 - y^2$, $x = t^2-1$, $y = 3\sin\pi t$

$$\frac{du}{dt} = ?$$

$u_x = 2x$

$u_y = -2y$

sol

$$\frac{du}{dt} = \frac{d}{dt}\big(u(t^2-1,\ 3\sin\pi t)\big)$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \nabla u(t^2-1, 3\sin\pi t) \bullet (x'(t), y'(t))$$

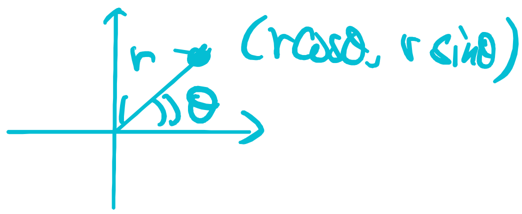
$$= u_x(t^2-1, 3\sin\pi t)\cdot 2t + u_y(t^2-1, 3\sin\pi t)\ 3\pi\cos\pi t$$

$= 2(t^2-1)\cdot 2t + (-2\cdot 3\sin\pi t)\cdot 3\pi\cos\pi t$ #

③ (Polar coordinates)

Let $u = u(x,y)$ be a smooth function on $\mathbb{R}^2$

Suppose
$$x = r\cos\theta$$
$$y = r\sin\theta$$



(a) $u(r,\theta) = u(r\cos\theta, r\sin\theta)$

$\frac{\partial u}{\partial r} = ?$ $\quad \frac{\partial u}{\partial \theta} = ?$

sol

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(u(r\cos\theta, r\sin\theta)\right)$$

$$\overset{\text{chain rule}}{=} \overbrace{u_x(r\cos\theta, r\sin\theta)\cdot \frac{\partial(r\cos\theta)}{\partial r} + u_y(r\cos\theta, r\sin\theta)\cdot\frac{\partial(r\sin\theta)}{\partial r}}^{\nabla u}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}\left(u(r\cos\theta, r\sin\theta)\right)$$

$$\begin{pmatrix}\frac{\partial u}{\partial r}\\ \frac{\partial u}{\partial \theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{pmatrix}$$

$$\overset{\text{chain rule}}{=} \frac{\partial u}{\partial x}\cdot\frac{\partial(r\cos\theta)}{\partial\theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial(r\sin\theta)}{\partial\theta}$$

$$= -\frac{\partial u}{\partial x}\cdot r\sin\theta + \frac{\partial u}{\partial y}\cdot r\cos\theta$$ #

(b) $\Delta u \overset{\text{"Laplacian"}}{:=} u_{xx} + u_{yy} = ?$ (Assume $r \neq 0$)

<u>Sol</u>

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\Big(u_x(r\cos\theta, r\sin\theta)\cdot\cos\theta + u_y(r\cos\theta, r\sin\theta)\ \sin\theta \Big)$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} u_{xx}(r\cos\theta, r\sin\theta)\cos\theta\cdot\cos\theta + u_{xy}(r\cos\theta, r\sin\theta)\cos\theta\cdot\sin\theta + u_{yx}(\ ,\ )\cdot\sin\theta\cos\theta + u_{yy}(\ \ )\cdot\sin\theta\sin\theta$$

$$\frac{\partial^2 u}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(-u_x\, r\sin\theta + u_y\, r\cos\theta\right)$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \cancel{-}(u_{xx}(-r\sin\theta) + u_{xy}\cdot r\cos\theta)\, r\sin\theta + (-u_x r\cos\theta) + (u_{yx}\cdot(-r\sin\theta) + u_{yy}\, r\cos\theta)\cdot r\cos\theta + u_y(-r\sin\theta)$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = u_{xx}\cdot\cos^2\theta + u_{yy}\sin^2\theta + u_{xx}\sin^2\theta + u_{yy}\cos^2\theta - \frac{1}{r}\underbrace{(u_x\cos\theta + u_y\sin\theta)}_{= \frac{\partial u}{\partial r}}$$

$\cos^2\theta + \sin^2\theta = 1$, $\sin^2\theta + \cos^2\theta = 1$

$$= u_{xx} + u_{yy} - \frac{1}{r}\frac{\partial u}{\partial r} \quad \cancel{= \Delta u}$$

#

④ (Implicit function)

Assume

$$u(x,y) = 2x^2y - y^3 + 1 - x - 2y = 0$$

$= u(\vec{\gamma}(x))$

Then $\dfrac{dy}{dx} = ?$

Sol

Let $\vec{\gamma}(x) = (x, y(x))$

$$\Rightarrow 0 = \frac{d}{dx} u(\vec{\gamma}(x)) \overset{\text{chain}}{\underset{\text{rule}}{=}} u_x \cdot \frac{dx}{dx} + u_y \cdot \frac{dy}{dx}$$

$$= (4xy - 1) \cdot 1 + (2x^2 - 3y^2 - 2) \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(4xy-1)}{2x^2 - 3y^2 - 2} \quad \#$$

更正 $x = r\cos\theta$, $y = r\sin\theta$

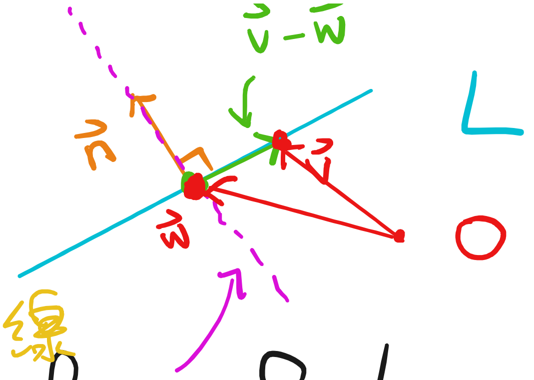
Let $u(x,y) = u(r\cos\theta, r\sin\theta)$.

$$\Delta u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r$$

Geometric meaning of gradient

$\mathbb{R}^2$

A normal vector (法向量) of a line L is a vector that is perpendicular to L:



法　线

The normal line of L is the line perpendicular to L.

That is, a vector $\vec n$ is normal to L if

$$\vec n \bullet (\vec v - \vec w) = 0 \quad \forall\, \vec v, \vec w \in L$$

If L is given by

$$ax + by = c,$$

$\underbrace{\vec v}_{=(v_1,v_2)}, \underbrace{\vec w}_{=(w_1,w_2)} \in L \iff$

$a v_1 + b v_2 = c$

$a w_1 + b w_2 = c$

then

$$(a, b) \bullet \underbrace{(\vec v - \vec w)}_{=(v_1-w_1,\, v_2-w_2)}$$

$$= a(v_1 - w_1) + b(v_2 - w_2)$$

$$= (\underbrace{a v_1 + b v_2}_{=c}) - (\underbrace{a w_1 + b w_2}_{=c}) = 0$$

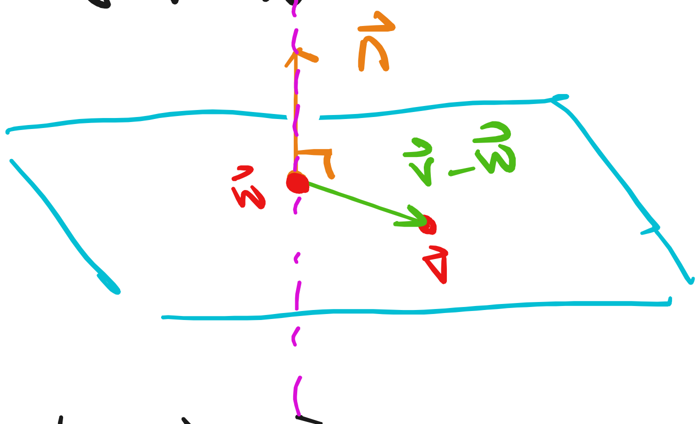
So the vector

$$\vec n = (a, b) \in \mathbb{R}^2$$

is a normal vector of $L: ax+by=c$

Similarly, a normal vector $\vec{n}$ of a plane P in $\mathbb{R}^3$ is a vector perpendicular to P, i.e.

$$\vec{n} \bullet (\vec{v} - \vec{w}) = 0 \quad \forall \vec{v}, \vec{w} \in P.$$

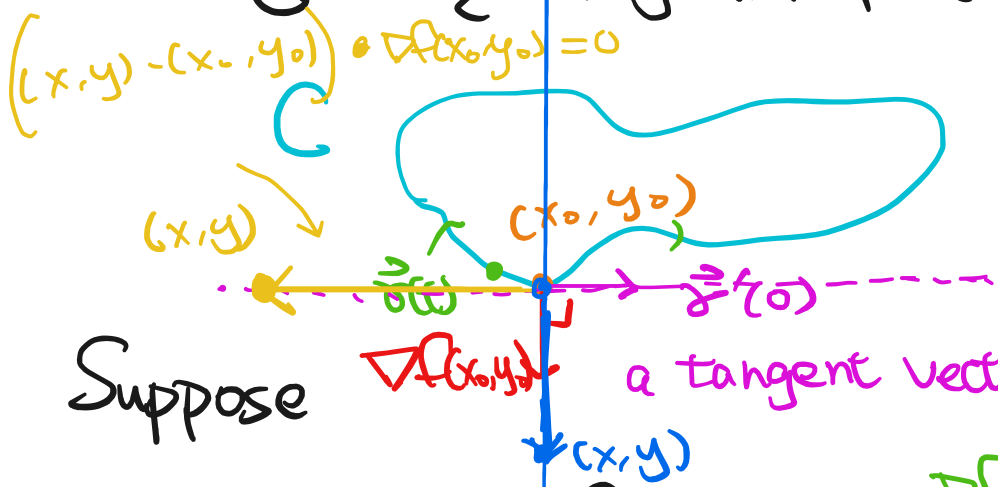


P: $ax+by+cz=d$

$\vec{n} = (a,b,c)$ is a normal vector of P.

Consider a curve in $\mathbb{R}^2$ given by

$$C = \{ (x,y) \in \mathbb{R}^2 \mid f(x,y) = c \}$$



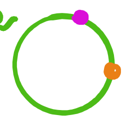
$f(x,y) = C$

Suppose $(x_0, y_0) \in C$ and $\nabla f(x_0,y_0) \neq \vec{0}$

e.g. $f(x,y) = x^2+y^2$

$c = 1$ $\quad \vec{\gamma}(t) = (\cos t, \sin t)$

$x^2+y^2=1$



$\nabla f = (2x, 2y)$

$\nabla f(1,0) = (2,0) \neq (0,0)$

Fact $\Rightarrow$ $\exists\ \vec{\gamma} : (-\varepsilon, \varepsilon) \to C$ s.t. $\vec{\gamma}(0) = (x_0, y_0)$,

$\vec{\gamma} = (\gamma_1(t), \gamma_2(t))$ $\qquad \vec{\gamma}'(0) \neq \vec{0}$

$$\Rightarrow \quad f(\gamma_1(t), \gamma_2(t)) = f(\vec{\gamma}(t)) = c$$

$$\Rightarrow \quad \left.\frac{d}{dt}\right|_{t=0} \left(f(\vec{\gamma}(t))\right) = \left.\frac{d}{dt}\right|_{t=0} (c) = 0$$

$\parallel$ chain rule

$$\nabla f(\vec{\gamma}(0)) \bullet \vec{\gamma}'(0)$$

$$\parallel$$

$$\nabla f(x_0, y_0) \bullet \vec{\gamma}'(0)$$

$\Rightarrow$ $\nabla f(x_0, y_0)$ is a normal vector of the tangent line of C at (x_0, y_0)

<u>Prop</u> (16.4.4, 16.4.5)

Suppose $\nabla f(x_0, y_0) \neq \vec{0}$. The tangent line of

$$C: \quad f(x,y) = c$$

at (x_0, y_0) is given by

$$\nabla f(x_0, y_0) \bullet (x - x_0, y - y_0) = 0$$

$$\parallel$$

$$(f_x(x_0, y_0))(x - x_0) + (f_y(x_0, y_0))(y - y_0) = 0$$

Furthermore, the <u>normal line</u> $=$ normal line of of C at

(x_0, y_0) is given by the condition

$$\nabla f(x_0,y_0) \parallel (x-x_0,\ y-y_0)$$

$\leftarrow$ parallel

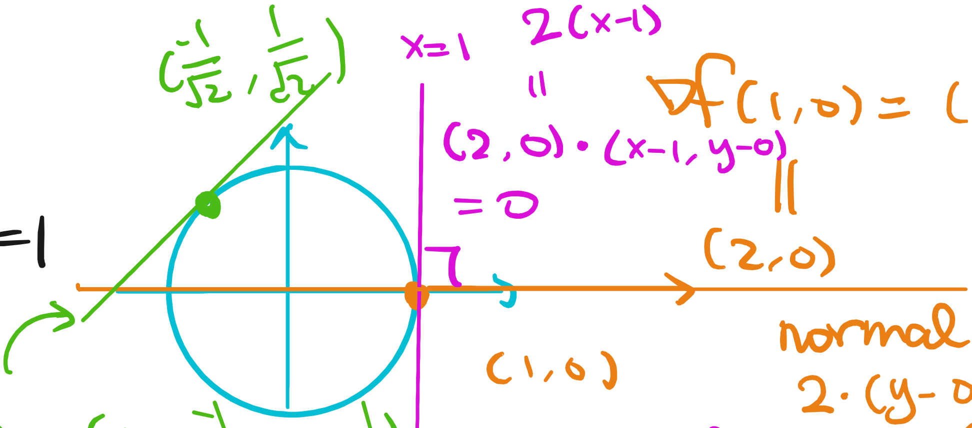
tangent line of C at (x_0, y_0)

$$\nabla f(x_0,y_0) = (f_x(x_0,y_0),\ f_y(x_0,y_0))$$

$$\Leftrightarrow\quad \big(f_x(x_0,y_0)\big)(y-y_0) = \big(f_y(x_0,y_0)\big)(x-x_0)$$

<u>Example</u>

$x^2+y^2=1$



$2(x-1) = (2,0)\cdot(x-1,\ y-0) = 0$

$\nabla f(1,0) = (2,0)$

normal line: $2\cdot(y-0) = 0(x-1) \Leftrightarrow y=0$

tangent line: $x=1$

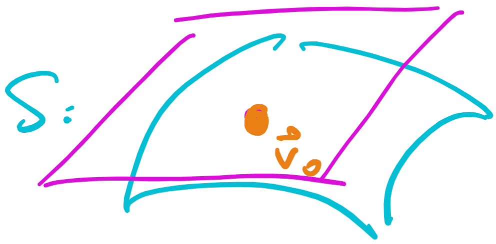
$$0 = \nabla f\left(\frac{-1}{\sqrt2}, \frac{1}{\sqrt2}\right)\cdot\left(x-\frac{-1}{\sqrt2},\ y-\frac{1}{\sqrt2}\right)$$

$$= (-\sqrt2, \sqrt2)\cdot\left(x+\frac{1}{\sqrt2},\ y-\frac{1}{\sqrt2}\right)$$

$\nabla f = (2x, 2y)$

$$= -\sqrt2 x - 1 + \sqrt2 y - 1 = -\sqrt2 x + \sqrt2 y - 2 = 0$$

$$\Leftrightarrow\quad x - y = -\frac{2}{\sqrt2} = -\sqrt2$$



Consider

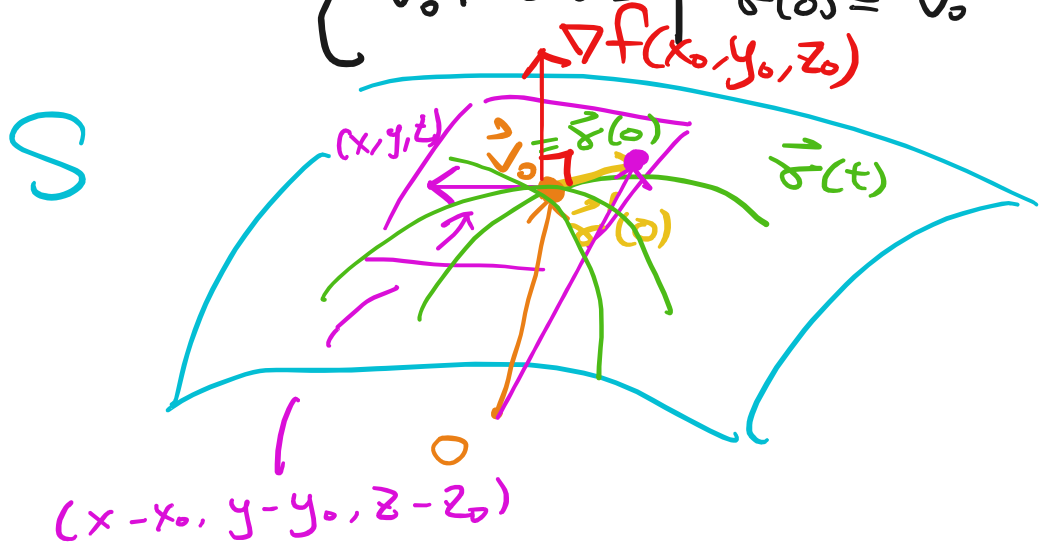
$$S = \{(x,y,z)\in\mathbb{R}^3 \mid f(x,y,z) = c\}$$

Suppose $\vec{v}_0 = (x_0, y_0, z_0) \in S$

and $\nabla f(\vec{v}_0) \neq \vec{0}$

The tangent plane of S at $\vec{v}_0$ is

$$\vec{v}_0 + T_{\vec{v}_0} S = \left\{ \vec{v}_0 + \vec{\gamma}'(0) \;\middle|\; \begin{array}{l} \vec{\gamma}: (-\varepsilon, \varepsilon) \to S \\ \vec{\gamma}(0) = \vec{v}_0 \end{array} \right\}$$



Since $\vec{\gamma}(t) \in S$, we have

$$f(\vec{\gamma}(t)) = c$$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} \left(f(\vec{\gamma}(t))\right) = \left.\frac{d}{dt}\right|_{t=0} (c) = 0$$

$\|$ chain rule $\qquad\qquad\qquad\qquad$ $\|$

$$\nabla f(\vec{\gamma}(0)) \bullet \vec{\gamma}'(0) = \nabla f(x_0, y_0, z_0) \bullet \vec{\gamma}'(0)$$

$\Rightarrow \nabla f(x_0, y_0, z_0)$ is a normal vector of $\vec{v}_0 + T_{\vec{v}_0} S$

Prop (16.4.8, 16.4.10)

Suppose $\nabla f(x_0, y_0, z_0) \neq \vec{0}$. Then the tangent plane of

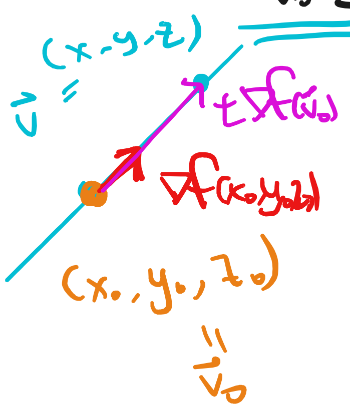
$$S: \quad f(x,y,z) = 0 \qquad\qquad \vec{v}_0 + T_{\vec{v}_0} S$$

at (x_0, y_0, z_0) is given by $\quad \|$

$$\nabla f(x_0, y_0, z_0) \bullet (x - x_0, y - y_0, z - z_0) = 0$$

Furthermore, the normal line of $\vec{v}_0 + T_{\vec{v}_0} S$ (or S) at $\vec{v}_0$ is given by

$\vec{v}_0 + t \nabla f(\vec{v}_0)$



$$\begin{cases} x = x_0 + t\, f_x(\vec{v}_0) \\ y = y_0 + t\, f_y(\vec{v}_0) \\ z = z_0 + t\, f_z(\vec{v}_0) \end{cases} \qquad t \in \mathbb{R}.$$

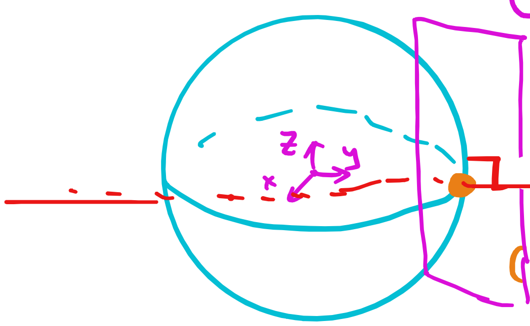
Example

$$\nabla f = (2x, 2y, 2z)$$

$$f(x,y,z) = x^2 + y^2 + z^2 = 1$$

$0 = (0, 2, 0) \bullet (x-0, y-1, z)$

$= 2(y-1) \Leftrightarrow y = 1$



$\nabla f(0,1,0) = (0,2,0)$

$(0,1,0)$

normal line:

$$\begin{cases} x = 0 + t \cdot 0 = 0 \\ y = 1 + t \cdot 2 = 1+2t \\ z = 0 + t \cdot 0 = 0 \end{cases}, \; t \in \mathbb{R}$$

$= y$-axis

<u>Example</u>

At what point(s) of the surface

$$z = 3xy - x^3 - y^3$$

is the tangent plane horizontal ? 水平的

i.e. "$z = c$" for some $c \in \mathbb{R}$

<u>Sol</u>:

The surface is given by

normal vector $= t\cdot(0,0,1)$

$$f(x,y,z) = 3xy - x^3 - y^3 - z = 0$$

Solve $\nabla f = (0,0,t)$

$$\left.\begin{array}{l} \nabla f = (0,0,t) \\ \parallel \\ (3y-3x^2,\ 3x-3y^2,\ -1) \end{array}\right) \Rightarrow t=-1, \ \begin{cases} y = x^2 \\ x = y^2 \end{cases}$$

$\Rightarrow x = x^4 \Rightarrow x = 0$ or 1

$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Downarrow z = 0 \quad \text{or} \quad \begin{cases} x = 1 \\ y = 1 \end{cases} \Downarrow z = 1$$

Extreme values

Def (Def 16.5.1)

We say $f = f(x_1, \dots, x_n)$ has a

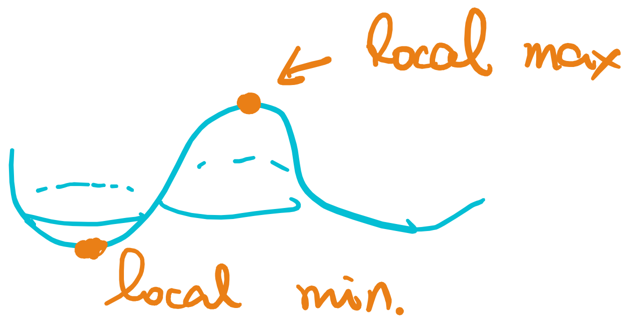
local minimum at $\vec{v}_0$ if $\exists \delta > 0$ s.t.

local maximum

$$f(\vec{v}_0) \leq f(\vec{v}) \quad \text{when} \quad \|\vec{v} - \vec{v}_0\| < \delta$$

$$f(\vec{v}_0) \geq f(\vec{v}) \quad \text{when} \quad \|\vec{v} - \vec{v}_0\| < \delta$$

$z = f(x, y)$



Local extreme value

= local maximum or local minimum

Thm (Thm 16.5.2)

If f has a local extreme value

at $\vec{v}_0$ then

$$\nabla f(\vec{v}_0) = \vec{0}$$

or $\nabla f(\vec{v}_0)$ does not exist.

pf: Next week.

Such $\vec{v}_0$ are called <u>critical points</u> of f

$\vec{v}_0$ is a <u>stationary point</u> $\Leftrightarrow$ $\nabla f(\vec{v}_0)=0$

$\vec{v}_0$ is a <u>saddle point</u> $\Leftrightarrow$ $\nabla f(\vec{v}_0)=0$ and f does not have an extreme value at $\vec{v}_0$.

馬安 點

Problem 2 Show that $\|\vec{\gamma}(t)\|$ is constant if and only if $\vec{\gamma}(t)\cdot\vec{\gamma}'(t)=0$ for all t.

常錯：

$\|\vec{\gamma}(t)\|=1\ \forall t$

1. $\|\vec{\gamma}(t)\|$ 是常數無法推得 $\vec{\gamma}(t)$ 是固定的一個點。例如：$\vec{\gamma}(t)=(\cos t,\sin t)$。

$\vec{\gamma}'(t)=(-\sin t,\cos t)$

2. $\vec{\gamma}(t)\cdot\vec{\gamma}'(t)=0$ 也無法推得 $\vec{\gamma}(t)=0$ 或 $\vec{\gamma}'(t)=0$。同一個例子可見得：$\vec{\gamma}(t)=(\cos t,\sin t)$。

$\vec{\gamma}(t)\cdot\vec{\gamma}'(t)$

3. 許多同學用到 $\frac{d}{dt}\|\vec{\gamma}(t)\|=\frac{\vec{\gamma}(t)\cdot\vec{\gamma}'(t)}{\|\vec{\gamma}(t)\|}$。這裡需要條件 $\|\gamma(t)\|\neq 0$ 對**所有** t。

$\|\vec{\gamma}(t)\|^2=\vec{\gamma}(t)\cdot\vec{\gamma}(t)$

$=\cos t(-\sin t)+\sin t\cos t$

   - 部分同學有分情況 $\|\vec{\gamma}(t)\|\neq 0$ 以及 $\|\vec{\gamma}(t)\|=0$ 討論的做法。但同學未注意到還有一種情況是 $\|\vec{\gamma}(t)\|$ 在某些時間 t 可能是 0。這種狀況下甚至 $\|\vec{\gamma}(t)\|$ 都不一定可以微分，例如 $\vec{\gamma}(t)=(t,t)$，在 $t=0$ 時 $\|\vec{\gamma}(t)\|$ 是不可微分的。

$=0$

4. 提醒同學，if and only if 是兩個方向都需證明。

Average	46.98214286
Average (except zero)	52.0990099
Quartile 0	0
Quartile 1	46.75
Quartile 2	54
Quartile 3	60
Quartile 4	60
標準差	18.38574759
非零數	101
不到50%人數	15
滿分人數	33