Calculus, Spring 2026, week 11

From now on, we will consider (real-valued) functions of several variables such as

$$f(x,y) = x^2+y^2$$

$$g(x,y,z) = x^2+y^2+z^2$$

§ Partial derivatives

Consider functions of 2 variables for simplicity.

Recall

$$\underset{\substack{\shortparallel \\ \frac{d\phi}{dx}}}{\phi'(x)} = \lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}$$

$\phi(x) = f(x,y)$

Def (Def. 15.4.1)

Let $f:\mathbb{R}^2\to\mathbb{R}$. The partial derivatives of f are defined as follows:

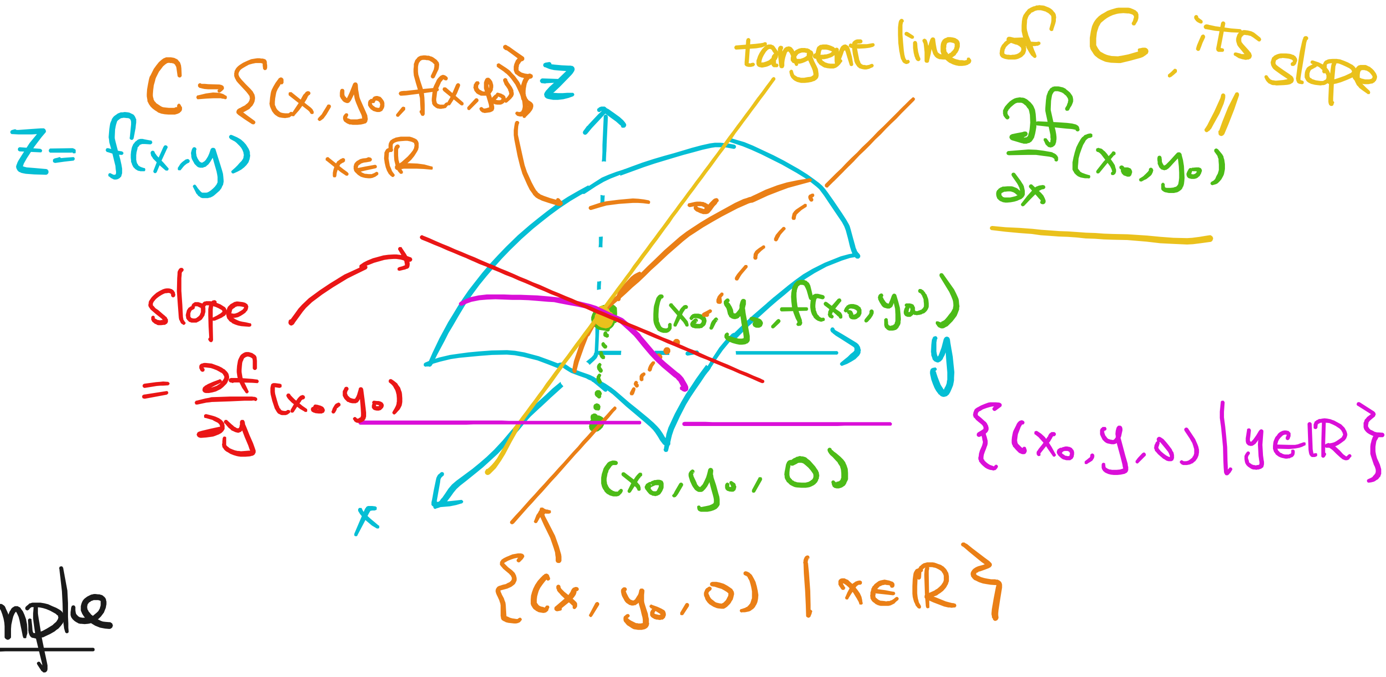
notations

$$\boxed{f_x}(x,y) = \boxed{\frac{\partial f}{\partial x}}(x,y) := \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

partial derivative of f with respect to x

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) := \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

provided these limits exist.



Example

Calculate f_x and f_y

① $f(x,y) = x^2+y^2$

$$\Rightarrow f_x = \frac{\partial f}{\partial x} = 2x + 0 = 2x$$

$$f_y = 0 + 2y = 2y$$ #

Recall: $\frac{d}{d\theta}\tan^{-1}\theta = \frac{1}{1+\theta^2}$

② $f(x,y) = x\tan^{-1}(xy)$

$$\Rightarrow f_x = \frac{d}{dx}(x)\cdot\tan^{-1}(xy) + x\cdot\frac{d}{dx}\tan^{-1}(xy)$$

$$= \tan^{-1}(xy) + x\cdot\frac{1}{1+(xy)^2}\cdot\boxed{\frac{\partial(xy)}{\partial x}} = y$$

$$= \tan^{-1}(xy) + \frac{xy}{1+(xy)^2}$$

$$f_y = x\cdot\frac{1}{1+(xy)^2}\cdot\overset{\frac{\partial(xy)}{\partial y}\ //}{\boxed{x}} = \frac{x^2}{1+(xy)^2}$$ #

③ $f(x,y) = e^{xy} + \ln(x^2+y)$

$$\Rightarrow f_x = e^{xy} \cdot \boxed{y}^{\,= \frac{\partial(xy)}{\partial x}} + \frac{1}{x^2+y}\,\boxed{2x} = \frac{\partial(x^2+y)}{\partial x}$$

$$f_y = e^{xy} \cdot \underset{\frac{\partial(xy)}{\partial y}}{\underset{\shortparallel}{\boxed{x}}} + \frac{1}{x^2+y} \quad \underset{\frac{\partial(x^2+y)}{\partial y}}{\underset{\shortparallel}{1}} \qquad \#$$

Def (Def 15.4.2)

Let $f : \mathbb{R}^3 \to \mathbb{R}$. $f = f(x,y,z)$.

Define

$$\frac{d\phi}{dx} = \underset{\shortparallel}{\phi'(x)} \quad \text{where} \quad \phi(x) = f(x,y,z)$$

$$f_x(x,y,z) = \frac{\partial f}{\partial x}(x,y,z) := \lim_{h \to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

the partial derivative of f with respect to x.

$$f_y(x,y,z) = \frac{\partial f}{\partial y}(x,y,z) := \lim_{h \to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}$$

$$f_z(x,y,z) = \frac{\partial f}{\partial z}(x,y,z) := \lim_{h \to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$. $f = f(x_1, x_2, \dots, x_n)$

then we define

$$f_{x_k}(x_1, \dots, x_n) = \frac{\partial f}{\partial x_k}(x_1, \dots, x_n)$$

$$:= \lim_{h \to 0} \frac{f(x_1, \dots, x_{k-1}, x_k + h, x_{k+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

Example

① $f(x,y,z) = xy^2z^3$

$\Rightarrow f_x = y^2z^3 \quad , \quad f_x(0,1,2) = 1^2 \cdot 2^3 = 8$

$f_y = x(2y)\cdot z^3$

$f_z = xy^2(3z^2)$

② $f(x,y,z,u,v) = xy^2 - e^{z+uv}$

$\Rightarrow f_x = y^2 - 0 = y^2$

$f_y = x \cdot 2y - 0 = 2xy$

$f_z = 0 - e^{z+uv} \cdot 1 = -e^{z+uv}$

$f_u = 0 - e^{z+uv} \cdot v = -v e^{z+uv}$

$f_v = 0 - e^{z+uv} \cdot u = -u e^{z+uv}$ #

Def (Higher order partial derivatives, p.782)

Let $f : \mathbb{R}^n \to \mathbb{R}$

A second order derivative of f is a partial derivative of a partial derivative of f

More generally, a <u>k-th order partial derivative</u> of f is a partial derivative of a (k-1)-th order partial derivative of f.

<u>Example</u>

Let $f(x,y,z) = xy^2z^3$. Compute the k-th order partial derivatives of f.

1st order:

$$f_x = \underline{y^2z^3}, \quad f_y = \underline{2xyz^3}, \quad f_z = \underline{3xy^2z^2}$$

2nd order:

$$f_{xx} = (f_x)_x = 0, \quad (f_x)_y = f_{xy} = \boxed{2yz^3}, \quad \overset{(f_x)_z}{\overset{\shortparallel}{f_{xz}}} = \underline{3y^2z^2}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad f_{xz} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yx} = (f_y)_x = \boxed{2yz^3}, \quad f_{yy} = 2xz^3, \quad f_{yz} = \underline{6xyz^2}$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

→ $f_{zx} = \underline{3y^2z^2}, \quad f_{zy} = \underline{6xyz^2}, \quad f_{zz} = 6xy^2z$

$$f_{zx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x \partial z}$$

3rd order:

$$f_{xyy} = 2z^3 = f_{yxy} = f_{yyx},$$

$$f_{xyz} = f_{xzy} = f_{yxz} = f_{yzx} = f_{zxy} = f_{zyx} = 6yz^2,$$

$$f_{xzz} = f_{zxz} = f_{zzx} = 6y^2z,$$

$$f_{yyz} = f_{yzy} = f_{zyy} = 6xz^2$$

$$f_{yzz} = f_{zyz} = f_{zzy} = 12xyz.$$

$f_{zzz} = 6xy^2$. and the others $= 0$.

<u>Remark</u>

From this computation, observe that

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \circledast$$

in this example.

In general, $\circledast$ holds under some <u>suitable assumptions</u> = ?

To describe the assumptions, we need the notions of continuity and limit for functions of several variables.

Def (§15.6)

Let $f:\mathbb{R}^2\to\mathbb{R}$ (in general, $f:\mathbb{R}^n\to\mathbb{R}$)

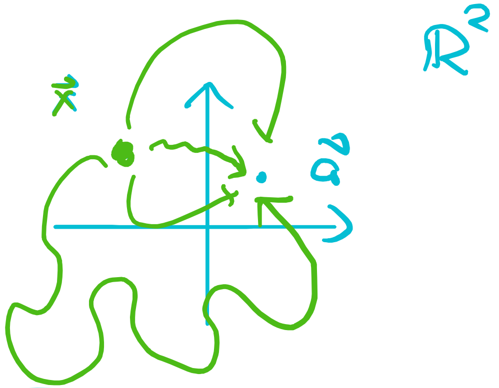
$$\overset{\|}{f(\vec{x})} = f(x_1,x_2)$$

$\vec{x} = (x_1, x_2)$

$\vec{a} = (a_1, a_2)$ $\in\mathbb{R}^2$

We say the limit

$$\lim_{\boxed{\vec{x}\to\vec{a}}} f(\vec{x}) \ (= L)$$

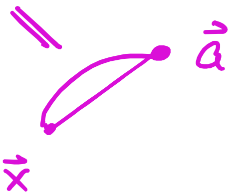


exists if $\exists L\in\mathbb{R}$ s.t. $\forall\varepsilon>0,\ \exists\delta>0$ s.t.

$$|f(\vec{x})-L|<\varepsilon$$

whenever $0<\underbrace{\|\vec{x}-\vec{a}\|}<\delta$

$$= \sqrt{(x_1-a_1)^2+(x_2-a_2)^2}$$



In this case, we write

$$\boxed{\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L}$$

Recall: $\lim_{x\to a} f(x) = L \iff \forall\varepsilon>0\ \exists\delta>0$ s.t. $|f(x)-L|<\varepsilon$ when $0<|x-a|<\delta$

Example

Let $f(x_1,x_2) = x_1\cdot x_2$

Then $\lim\limits_{(x_1,x_2)\to(0,0)} f(x_1,x_2) = 0$

pf

$\forall \varepsilon > 0,\ \exists\, \delta\ (= \sqrt{\varepsilon}) > 0$ s.t. $\Rightarrow \begin{cases} |x_1| < \delta \\ |x_2| < \delta \end{cases}$

if $0 < \underbrace{\|(x_1,x_2) - (0,0)\|}_{=\sqrt{x_1^2 + x_2^2}} < \delta$ $\quad \geq \sqrt{x_1^2} = |x_1|$, $\quad \geq \sqrt{x_2^2} = |x_2|$

then $|f(x_1,x_2) - 0| = |x_1 x_2 - 0|$

$= |x_1|\cdot|x_2| < \sqrt{\varepsilon}\cdot\sqrt{\varepsilon} = \varepsilon$

$\Rightarrow \lim\limits_{(x_1,x_2)\to(0,0)} f(x_1,x_2) = 0$ #

Def

A function $f: \mathbb{R}^2 \to \mathbb{R}$ (or $f: \mathbb{R}^n \to \mathbb{R}$) is continuous at $\vec{x} = \vec{a}$ if

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$$

(or $\mathbb{R}^n$)

We say that f is continuous on $\mathbb{R}^2$ if f is continuous at each point in $\mathbb{R}^2$.
(or $\mathbb{R}^n$)

Example

Following functions are some (smooth) Continuous functions on $\mathbb{R}^2$

① Constant functions

② Polynomials of two variables

e.g. x, xy, xy^2, $x+xy+x^3-xy^4$

③ $e^{P(x,y)}$, $\sin P(x,y)$, $\cos P(x,y)$

($P(x,y)$ = a polynomial of 2 variables)

and combination of them (addition, scalar product, multiplication composition)

e.g. $\sin\left(e^{e^x\cdot\cos y+e^{xy}+x^2+y^2}\right)$ is continuous on $\mathbb{R}^2$

Thm (Page 783)

Let $f=f(x,y)$ be a real-valued function of 2 variables. Suppose f, f_x, f_y, f_{xy}, f_{yx} are continuous on $\mathbb{R}^2$.

Remark:

Apply Thm to

$f_{xyz}=(f_x)_{yz}\overset{?}{=}(f_x)_{zy}$

under some assumption

Then

$$f_{xy} = f_{yx} \quad \text{on} \quad \mathbb{R}^2$$

pf: skip.

§ Gradient

For convenience, unless otherwise stated, we will assume that the functions in discussion are smooth, i.e. all the partial derivative of any order are continuous on $\mathbb{R}^n$

Def (Def 16.1.2, Thm 16.1.3)

Let f be a real-valued function of 3 variable (in general, n variables). The 梯度 gradient of f at $\vec{p} \in \mathbb{R}^3$ is the vector

$$\nabla f(\vec{p}) := \left(f_{x_1}(\vec{p}),\ f_{x_2}(\vec{p}),\ f_{x_3}(\vec{p})\right) \in \mathbb{R}^3$$

(If $f = f(x_1, \cdots, x_n)$, then $p \in \mathbb{R}^n$

$$\nabla f\big|_p = \left(f_{x_1}\big|_p, f_{x_2}\big|_p, \cdots, f_{x_n}\big|_p\right) \in \mathbb{R}^n \quad)$$

Example

① Let

$$f(x,y) = e^y - ye^x$$

Then

$$f_x = 0 - ye^x, \qquad f_y = e^y - e^x$$

$$\Rightarrow \quad \nabla f = (-ye^x, \; e^y - e^x) \quad \in \mathbb{R}^2$$

$$\nabla f(0,1) = (-1 \cdot e^0, \; e^1 - e^0)$$

$$= (-1, \; e-1) \in \mathbb{R}^2 \quad \#$$

② Let $g(x,y,z) = x\sin(\pi y) + y\cos(z)$

Then

$$g_x = \sin(\pi y), \quad g_y = x(\cos \pi y) \cdot \pi + \cos z$$

$$g_z = -y\sin z$$

$$\Rightarrow \quad \nabla g = (\sin(\pi y), \; \pi x\cos(\pi y) + \cos z, \; -y\sin z) \in \mathbb{R}^3 \quad \#$$

Thm (16.2.1)

Let f, g be smooth functions, $\alpha \in \mathbb{R}$.

(i) $\nabla(f(\vec{x}) + g(\vec{x})) \underset{\uparrow}{=} \nabla f(\vec{x}) + \nabla g(\vec{x})$

$$\frac{\partial}{\partial x_i}(f+g) = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}$$

(ii) $\nabla(\alpha f(\vec{x})) \underset{\uparrow}{=} \alpha \cdot \nabla f(\vec{x})$

$$\frac{\partial}{\partial x_i}\left(\alpha f(\vec{x})\right) = \alpha \frac{\partial f}{\partial x_i}(\vec{x})$$

(iii) $$\nabla\left(f(\vec{x})\, g(\vec{x})\right) = g(\vec{x}) \cdot \nabla f(\vec{x}) + f(\vec{x}) \cdot \nabla g(\vec{x})$$

$$\frac{\partial}{\partial x_i}(f \cdot g) = \frac{\partial f}{\partial x_i} \cdot g + f \cdot \frac{\partial g}{\partial x_i}$$

$$\left(\frac{\partial (fg)}{\partial x_1}, \dots, \frac{\partial (fg)}{\partial x_n}\right) = \left(\frac{\partial f}{\partial x_1} g + f \cdot \frac{\partial g}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} g + f \frac{\partial g}{\partial x_n}\right)$$

$$= g \cdot (f_{x_1}, \dots, f_{x_n}) + f \cdot (g_{x_1}, \dots, g_{x_n})$$

Def (Def 16.2.2)

Let $\vec{u}$ be a unit vector, i.e. $\|\vec{u}\| = 1$.

The limit

$$\left(f'_{\vec{u}} =\right) \nabla_{\vec{u}} f(\vec{x}) := \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h},$$

notation in textbook

if exists, is called the directional derivative of f at $\vec{x}$ in the direction $\vec{u}$.

方向 導數

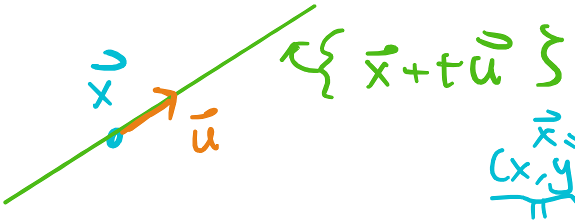
Remark

① The assumption "$\|\vec{u}\| = 1$" is not important for the limit $\nabla_{\vec{u}} f(\vec{x}) = \lim\limits_{h \to 0} \dfrac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$

We impose this assumption "$\|\vec{u}\| = 1$" because a "direction" is conventionally represented by a unit vector.

② $\nabla_{\vec u} f(\vec x)$ = rate of change of f in the direction $\vec u$, and

$$\nabla_{\vec u} f(\vec x) = \frac{d}{dt} f(\vec x + t\vec u)\Big|_{t=0}$$



$$\overset{\vec x =}{\underline{(x,y)}} + h\,\overset{\vec e_1}{\underline{(1,0)}} \qquad \|\vec e_1\| = 1$$

③
$$f_x(x,y) \overset{\text{def}}{=} \lim_{h\to 0} \frac{f\boxed{(x+h,y)} - f(x,y)}{h}$$
$$= \lim_{h\to 0} \frac{f(\vec x + h\vec e_1) - f(\vec x)}{h}$$
$$= \nabla_{\vec e_1} f(x,y)$$

More generally

$$\vec e_i = (0,\cdots,0,\overset{i\text{-th}}{\overset{\downarrow}{1}},0,\cdots0)$$

$$(x_1,\cdots,x_{i-1},\ x_i+h,\ x_{i+1},\cdots,x_n) = \vec x + h\vec e_i$$

$$\frac{\partial f}{\partial x_i}(\vec x) = \lim_{h\to 0} \frac{f\boxed{(\vec x + h\vec e_i)} - f(\vec x)}{h}$$
$$= \nabla_{\vec e_i} f(\vec x).$$

Note that $\vec u$ can be $\left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right) \neq (1,0)$, $\neq (0,1)$

$\exists\ \infty$ many directions

Thm (Thm 16.2.4)

$$\nabla_{\vec u} f(\vec x) = Df(\vec x)\cdot \vec u$$

Pf $\vec{u}=(u_1,u_2)$, $\vec{x}=(x_1,x_2)$

We prove it for the case of $\underline{\underline{2}}$ variables

The case of n variables is similar.

$$f(\vec{x}+h\vec{u})-f(\vec{x})=f(x_1+hu_1,\ x_2+hu_2)-f(x_1,x_2)$$

view it as a fixed number

$$=f(x_1+hu_1,\underline{x_2+hu_2})-f(x_1,\underline{x_2+hu_2})$$

MVT

$$=hu_1\cdot f_{x_1}(c_h,x_2+hu_2)$$

$$+f(x_1,x_2+hu_2)-f(x_1,x_2)$$

By Mean Value Thm (apply to the function $f(-,x_2+hu_2)$)

$\exists\ c_h$ between x_1 and x_1+hu_1 s.t.

$$f(x_1+hu_1,x_2+hu_2)-f(x_1,x_2+hu_2)$$

$$=(\cancel{x_1}+\underline{\underline{hu_1}}-\cancel{x_1})\cdot f_{x_1}(c_h,x_2+hu_2)$$

By MVT (apply to $f(x_1,-)$)

$\exists\ c_h'$ between x_2 and x_2+hu_2 s.t.

$$f(x_1,x_2+hu_2)-f(x_1,x_2)$$

$$=(x_2+hu_2-x_2)\cdot f_{x_2}(x_1,c_h')$$

MVT: If $f\in C'$
then
$f(x)-f(y)$
$=(x-y)f'(c)$
for some
c between
x and y

So

$$\nabla_{\vec{u}}f(\vec{x})=\lim_{h\to 0}\frac{f(\vec{x}+h\vec{u})-f(\vec{x})}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h} u_1 \cdot f_{x_1}(c_h, x_2+hu_2) + \cancel{h} u_2 f_{x_2}(x_1, c_h')}{\cancel{h}}$$

$$= \lim_{h \to 0} u_1 \cdot f_{x_1}\Big(\underset{\overset{\downarrow}{x_1}}{\overset{\text{between } x_1 \text{ and } x_1+hu_1}{\boxed{c_h}}}, \underbrace{x_2+hu_2}_{x_2}\Big) + u_2 f_{x_2}\Big(x_1, \underset{\overset{\swarrow}{x_2}}{\overset{\text{between } x_2 \text{ and } x_2+hu_2}{\boxed{c_h'}}}\Big)$$

f_{x_1}, f_{x_2} are continuous (f is smooth)

$$= u_1 \cdot f_{x_1}(x_1, x_2) + u_2 \cdot f_{x_2}(x_1, x_2)$$

$$= \nabla f(\vec{x}) \bullet \vec{u} \qquad \#$$

Remark

$$\frac{d\, f(\vec{x}+t\vec{u})}{dt} = \frac{d}{ds}\Big|_{s=0} \Big(f\big(\vec{x} + \underbrace{(t+s)\vec{u}}_{}\big) \Big) \qquad (\vec{x}+t\vec{u})+s\vec{u}$$

$$= \nabla_{\vec{u}} f(\vec{x}+t\vec{u})$$

$$= \nabla f(\vec{x}+t\vec{u}) \bullet \vec{u}$$

Example

Let $f(x,y) = x^2+y^2$, $\vec{u} = (\frac{3}{5}, \frac{4}{5})$

Then

$$\nabla_{\vec{u}} f = \nabla f \bullet \vec{u}$$

$$= (2x, 2y) \bullet (\tfrac{3}{5}, \tfrac{4}{5})$$

$$= \frac{6}{5}x + \frac{8}{5}y$$

$$\nabla_{\vec{u}} f\,(1,2) = \frac{6}{5} + \frac{16}{5} \qquad \#$$

1. 上學期的極限。會的就會，不會的還是沒搞懂。
2. 一堆人以為級數收斂會蘊含 ratio 或 root 的極限 <1，但此為級數收斂的充分而非必要條件。
3. 前兩小題還好，第三小題很多人直接用 alternating seires test (Leibniz test)，當然是沒辦法直接用。
4. 幾乎都會寫，少部分在原點打開級數。
5. 很多人算收斂半徑會有以下問題：

   (i). 取了 root 的上極限（ratio 的極限）後忘記倒數。

   (ii). 有人這樣算

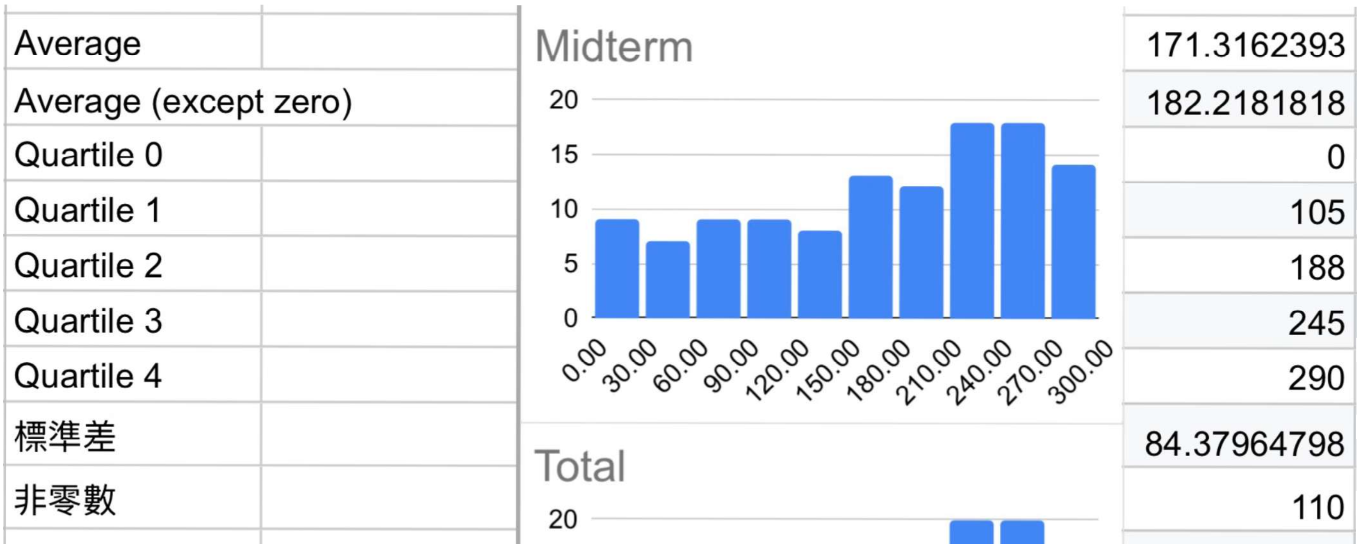
   $$R = \limsup_{n\to\infty} \frac{1}{|a_n|^{\frac{1}{n}}}$$

   。這當然不行，因為 limit superior 不會保持四則運算的關係。

   (iii). 還是很多人忘記處理邊界斂散性。
6. (i). 有少部分人沒有發現 $-\frac{1}{x}$ 會抵消掉，所以最終的式子保留 $-\frac{1}{x}$，並沒有整理乾淨。

   (ii). 或者更進一步的把其中一個 $\frac{1}{x}$ 在 $x=1$ 展開，所以式子包含 $\frac{1}{x}, x^k$ 和 $(1-x)^k$ 且收斂區間較小。
7. 有部分人不會寫，少部分人在最後計算時忘記乘上 π。
8. (b). Parseval’s identity 中的 $||f(x)||^2$ 記成 $|f(x)|^2$ 。
9. (a)(b). 有些人會直接把 p, q 當成非負整數，所以直接寫

   $$\Gamma(p) = (p-1)!。$$



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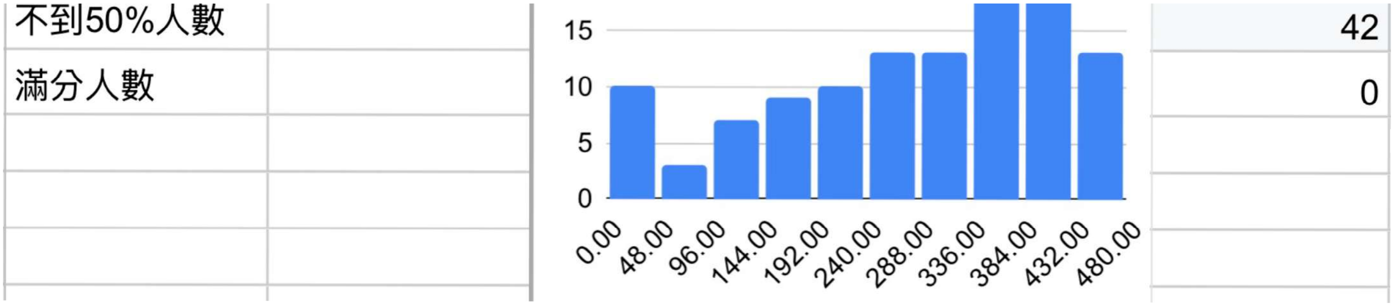
Average (except zero)		182.2181818
Quartile 0		0
Quartile 1		105
Quartile 2		188
Quartile 3		245
Quartile 4		290
標準差		84.37964798
非零數		110



不到50%人數		42
滿分人數		0

National Tsing Hua University

Calculus (II) – Final exam

Instructor: Hsuan-Yi Liao

Spring, 2026

Name: Answer

Student ID: ________________________________

- This exam contains 9 pages (including this cover page) and 9 questions.
- Total of points is 300.
- Time limit: **100 minutes**.
- Write down your computation or arguments in details unless otherwise stated.
- The use of a calculator, cell phone, or any other electronic device is **NOT** permitted.
- The use of books or notes of any kind is **NOT** permitted.

Distribution of Marks

Question	Points	Score
1	40	
2	20	
3	30	
4	30	
5	40	
6	30	
7	30	
8	40	
9	40	
Total:	300	

1. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences. Suppose that $\lim\limits_{n\to\infty} b_n \neq 0$.
   - (a) (20 points) Prove that there exists M such that $b_n \neq 0$ for any $n \geq M$.
   - (b) (20 points) Prove that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$.

(a)

Let $\beta = \lim\limits_{n\to\infty} b_n$ and $\varepsilon = \frac{|\beta|}{2}$. Since $\beta \neq 0$, $\varepsilon > 0$.

$\Rightarrow \exists N_\varepsilon$ s.t.

$$|\beta| - |b_n| \leq |b_n - \beta| < \varepsilon = \frac{|\beta|}{2} \qquad \forall n \geq N_\varepsilon$$

$$\Rightarrow \qquad |b_n| > |\beta| - \varepsilon = \frac{|\beta|}{2} > 0 \qquad \forall n \geq N_\varepsilon$$

Choose $M = N_\varepsilon \Rightarrow b_n \neq 0 \ \forall n \geq M$. #

(b) Let $\alpha = \lim\limits_{n\to\infty} a_n$, $\beta = \lim\limits_{n\to\infty} b_n$

$$\left| \frac{a_n}{b_n} - \frac{\alpha}{\beta} \right| = \frac{|\beta a_n - \alpha b_n|}{|\beta||b_n|}$$

By (a), $\exists M$ s.t. $|b_n| > \frac{|\beta|}{2} \quad \forall n \geq M$. $\Rightarrow |\beta||b_n| > \frac{\beta^2}{2} \ \forall n \geq M$

Since $\lim\limits_{n\to\infty} a_n = \alpha$, $\lim\limits_{n\to\infty} b_n = \beta$, $\forall \varepsilon > 0 \ \exists N_1, N_2$ s.t.

$$|a_n - \alpha| < \varepsilon \cdot \frac{|\beta|}{4} \qquad \forall n \geq N_1$$

$$|b_n - \beta| < \varepsilon \cdot \frac{\beta^2}{4(|\alpha|+1)} \qquad \forall n \geq N_2$$

Take $\tilde{N} = \max\{N_1, N_2, M\}$

$$\Rightarrow \frac{|\beta a_n - \alpha b_n|}{|\beta||b_n|} \leq \frac{|\beta a_n - \beta\alpha + \beta\alpha - \alpha b_n|}{\frac{|\beta|^2}{2}}$$

$$\leq \frac{|\beta||a_n - \alpha|}{\frac{|\beta|^2}{2}} + \frac{|\alpha||b_n - \beta|}{\frac{|\beta|^2}{2}}$$

$$< \frac{2}{\cancel{|\beta|}} \cdot \frac{\cancel{|\beta|}}{\cancel{4}} \cdot \varepsilon + \frac{\cancel{2}|\alpha|}{\cancel{\beta^2}} \cdot \frac{\cancel{\beta^2}}{2\cancel{4}(|\alpha|+1)} \varepsilon < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
$\frac{1}{2}$

$\forall n \geq \tilde{N}$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta}$$ #

2. (20 points) Suppose that the series $\sum_{k=1}^{\infty} |a_k|$ is convergent. Prove that the series $\sum_{k=1}^{\infty} a_k^2$ is convergent.

$\sum_{k=1}^{\infty} |a_k|$ Converges $\Rightarrow \exists M>0$ s.t. $\sum_{k=1}^{n} |a_k| \leq M \quad \forall n \in \mathbb{N}$

$\Rightarrow \quad \sum_{k=1}^{n} a_k^2 \leq \left(\sum_{k=1}^{n} |a_k|\right) \cdot \left(\sum_{k=1}^{n} |a_k|\right) \leq M^2 \quad \forall n \in \mathbb{N}$

$\Rightarrow \quad \sum_{k=1}^{\infty} a_k^2$ converges. #

$\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} < 1$

$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} < 1$

3. Does the series converge absolutely, converge conditionally, or diverge? Explain your reasoning. (Here, [x] denotes the greatest integer less than or equal to x.)

(a) (10 points) $\sum_{k=1}^{\infty} (-1)^k k \sin(1/k)$.

(b) (10 points) $\sum_{k=1}^{\infty} \frac{\sin(\pi k/2)}{k\sqrt{k}}$.

(c) (10 points) $\sum_{k=1}^{\infty} \frac{(-1)^{[k/3]}}{k}$.

(a) $\lim_{k\to\infty} k \cdot \sin(\frac{1}{k}) = \lim_{k\to\infty} \frac{\sin\frac{1}{k}}{\frac{1}{k}} = 1 \neq 0$

$\Rightarrow \lim_{k\to\infty} (-1)^k k \sin\frac{1}{k} \neq 0 \Rightarrow$ diverges #

$= 1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1, 1, 1, 1, \ldots$

(b) $\left|\frac{\sin(\frac{\pi k}{2})}{k\sqrt{k}}\right| \leq \frac{1}{k^{\frac{3}{2}}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges

$\sum_{i=0}^{\infty} \frac{(-1)^i}{3i+1} + \sum_{i=0}^{\infty} \frac{(-1)^i}{3i+2} + \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{3i+3}$

$\Rightarrow \sum_{k=1}^{\infty} \frac{\sin(\frac{\pi k}{2})}{k\sqrt{k}}$ absolutely converges. #

(c) Since $\left|\sum_{k=1}^{n} (-1)^{[\frac{k}{3}]}\right|$ is bounded by 3,

$\lim_{k\to\infty} \frac{1}{k} = 0$ and $(\frac{1}{k})_{k=1}^{\infty}$ is decreasing

by Dirichlet Test, $\sum_{k=1}^{\infty} \frac{(-1)^{[\frac{k}{3}]}}{k}$ is Conditionally Converges #

and $\sum \left|\frac{(-1)^{[\frac{k}{3}]}}{k}\right| = \sum \frac{1}{k}$ diverges.

4. Let $f(x) = x\sin x$.

   (a) (15 points) Find the Taylor expansion of $f(x)$ at $\frac{\pi}{2}$.

   (b) (15 points) Use the Taylor expansion of $f(x)$ to find the value of $f^{(100)}(\frac{\pi}{2})$.

(a) Recall $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \forall x \in \mathbb{R}$.

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\Rightarrow x\sin x = (x - \tfrac{\pi}{2} + \tfrac{\pi}{2}) \cdot \sin(x - \tfrac{\pi}{2} + \tfrac{\pi}{2})$$

$$= (x-\tfrac{\pi}{2}) \cdot \cos(x-\tfrac{\pi}{2}) + \tfrac{\pi}{2} \cos(x-\tfrac{\pi}{2})$$

$$= (x-\tfrac{\pi}{2}) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x-\tfrac{\pi}{2})^{2k} + \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x-\tfrac{\pi}{2})^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)} (x-\tfrac{\pi}{2})^{2k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{\pi}{2} (x-\tfrac{\pi}{2})^{2k} \quad \#$$

(b) $\frac{f^{(100)}(\frac{\pi}{2})}{100!}$ = coefficient of $(x-\frac{\pi}{2})^{100}$ in Taylor expansion at $\frac{\pi}{2}$

$$= \frac{(-1)^{50}}{(2 \cdot 50)!} \cdot \frac{\pi}{2}$$

$$\Rightarrow f^{(100)}(\tfrac{\pi}{2}) = \frac{\pi}{2} \quad \#$$

5. Find the interval of convergence.

(a) (20 points) $\sum_{k=1}^{\infty}(-1)^k\frac{k!}{k^3}(x-1)^k$.

(b) (20 points) $\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k(x+2)^k$.

(a) $$\left|\frac{(-1)^{k+1}\frac{(k+1)!}{(k+1)^3}}{(-1)^k\frac{k!}{k^3}}\right| = (k+1)\cdot\left(\frac{k}{k+1}\right)^3$$

Ratio Test $\to +\infty$ as $k\to\infty$

$$\frac{1}{\overline{\lim_{k\to\infty}}\sqrt[k]{|a_k|}} \neq \overline{\lim_{k\to\infty}}\frac{1}{\sqrt[k]{|a_k|}}$$

$\Rightarrow$ radius of conv. $= 0$

$\Rightarrow$ interval of conv $= \{1\}$. #

(b) $$\sqrt[k]{\left(1+\frac{1}{k}\right)^k} = \left(1+\frac{1}{k}\right) \to 1 \text{ as } k\to\infty$$

Root Test

$\Rightarrow$ radius of conv. $= 1$

At $x=-1$:

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k\cdot(1)^k = \sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k \text{ diverges since}$$

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e \neq 0$$

At $x=-3$:

$\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^k(-1)^k$ diverges since $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k(-1)^k \neq 0$

(diverges)

Conclusion:

interval of conv. $= (-3,-1)$. #

6. Set $f(x) = \dfrac{e^x - 1}{x}$.

(a) (15 points) Expand $f(x)$ in a power series.

(b) (15 points) Differentiate the series and prove that

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$$

$\leftarrow \sum_{k=0}^{\infty} a_k x^k$

$\forall x \in \mathbb{R}$

(a) $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \boxed{\frac{e^x-1}{x}} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} \quad \#$$

$$\leftarrow \frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{k=0}^{\infty} (1-x)^k$$

for $|1-x| < 1$

(b) By a,

$$\left(\frac{e^x-1}{x}\right)' = \sum_{k=1}^{\infty} \frac{k}{(k+1)!} x^{k-1}$$

$$\|$$

$$\frac{e^x \cdot x - (e^x-1)\cdot 1}{x^2} = \frac{xe^x - e^x + 1}{x^2}$$

at $x=1$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{k}{(k+1)!} = \frac{1\cdot e^1 - e^1 + 1}{1^2} = 1 \quad \#$$

$\sum_{k=1}^{\infty} a_k$ converges $\Rightarrow$ $\lim_{k\to\infty} a_k = 0$

$$\not\Leftarrow$$

$\sum_{k=1}^{\infty} a_k$ diverges $\Leftarrow$ $\lim_{k\to\infty} a_k \neq 0$

7. (30 points) Let $f(x) = \cos^6 x$. Define

$$F(c_0, c_1, c_2, d_1, d_2) = \int_{-\pi}^{\pi} \left| f(x) - \frac{c_0}{2} - c_1 \cos x - d_1 \sin x - c_2 \cos 2x - d_2 \sin 2x \right|^2 dx.$$

Find the minimum value of F. (Hint: You may use the identities $\cos(4x) = 8\cos^4 x - 8\cos^2 x + 1$ and $\cos(6x) = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$.)

$\cos 2x = 2\cos^2 x - 1$

$$\cos^6 x = \frac{1}{32}\left(\cos 6x + 48\cos^4 x - 18\cos^2 x + 1\right)$$

$$= \frac{1}{32}\left(\cos 6x + \frac{\cancel{48}^{\,6}}{\cancel{8}}\left(\cos 4x + 8\cos^2 x - 1\right) - 18\cos^2 x + 1\right)$$

$$= \frac{1}{32}\left(\cos 6x + 6\cos 4x \overset{15(\cos 2x + 1)}{\overset{\parallel}{\boxed{+30\cos^2 x}}} - 5\right)$$

$$= \frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x + \frac{15}{32}\cos 2x + \frac{5}{16}$$

$\Rightarrow$ The minimum of F occurs when

$$c_0 = \frac{5}{8}, \quad c_1 = d_1 = d_2 = 0, \quad c_2 = \frac{15}{32}$$

and min of F $= \displaystyle\int_{-\pi}^{\pi} \left| \frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x \right|^2 dx$

$$= \pi\left(\left(\frac{1}{32}\right)^2 + \left(\frac{3}{16}\right)^2\right) \quad \#$$

Recall (Parseval's identity)

If $f(x) \sim \dfrac{a_0}{2} + \displaystyle\sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$

then

$$\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

$\ast$

$|f(x)|^2$

8. Let $f(x)$ be the periodic function with period 2π such that $f(x) = x^2$ for $x \in (-\pi, \pi]$.

(a) (20 points) Find the Fourier series of $f(x)$.

(b) (20 points) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

(a)

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos kx \, dx = \frac{1}{\pi}\left(\overset{=0}{\cancel{x^2 \frac{\sin kx}{k}\Big|_{x=-\pi}^{x=\pi}}} - \int_{-\pi}^{\pi} 2x \frac{\sin kx}{k} dx \right)$$

$$= \frac{1}{\pi}\left(+2x \cdot \frac{\cos kx}{k^2}\Big|_{-\pi}^{\pi} - \underbrace{\int_{-\pi}^{\pi} 2 \frac{\cos kx}{k^2} dx}_{= 2 \frac{\sin kx}{k^3}\big|_{-\pi}^{\pi} = 0} \right)$$

$$= 4 \frac{(-1)^k}{k^2}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2}{3}\pi^2$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2 \sin kx \, dx}_{\text{odd}} = 0$$

$$\Rightarrow \quad f(x) \sim \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos kx$$

#

(b)

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

$$= \frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 + \sum_{k=1}^{\infty} \frac{16}{k^4}.$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx$$

$$= \frac{1}{\pi} \cdot 2 \cdot \frac{\pi^5}{5} = \frac{2}{5}\pi^4$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{1}{16}\left(\frac{2}{5} - \frac{2}{9}\right)\pi^4 = \frac{\pi^4}{90}$$

#

$$\frac{18-10}{45} = \frac{8}{45}$$

9. Recall that for $x, y > 0$, the Gamma function is

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$$

and the Beta function $B(x, y)$ is defined as:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

$\Gamma(x+1) = x\,\Gamma(x)$

$= (x-1)!$

$\uparrow$

only when $x \in \mathbb{N}$.

Prove the following properties:

(a) (10 points) For $p, q > 0$, show that $B(p, q) = B(p+1, q) + B(p, q+1)$.

(b) (10 points) For $p > 0$ and $n \in \mathbb{N}$, show that

$$B(p, n+1) = \frac{n!}{p(p+1)\cdots(p+n)}.$$

(c) (20 points) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(a) $B(p+1, q) = \dfrac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1)} = \dfrac{p\,\Gamma(p)\Gamma(q)}{(p+q)\,\Gamma(p+q)}$

$B(p, q+1) = \dfrac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)} = \dfrac{q}{(p+q)}\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

$\Rightarrow B(p+1, q) + B(p, q+1) = B(p, q)$ #

(b) $B(p, n+1) = \dfrac{\Gamma(p)\,\Gamma(n+1)}{\Gamma(p+n+1)} = \dfrac{\Gamma(p)\cdot n!}{(p+n)(p+n-1)\cdots p\;\Gamma(p)} = \dfrac{n!}{(p+n)\cdots p}$ #

(c) $B(\frac{1}{2}, \frac{1}{2}) = \displaystyle\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{1}{2}-1}\, dt \overset{t = \sin^2\theta \;\Rightarrow\; dt = 2\sin\theta\cos\theta\, d\theta}{=\!=} \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{-\frac{1}{2}} (\cos^2\theta)^{-\frac{1}{2}}\, 2\sin\theta\cos\theta\, d\theta$

$\quad\shortparallel$

$\dfrac{\Gamma(\frac{1}{2})\cdot\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \left(\Gamma(\tfrac{1}{2})\right)^2 \qquad = \displaystyle\int_0^{\frac{\pi}{2}} 2\, d\theta = \pi$

$\Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$ #