Calculus, Spring 2026, week 1

Infinite series 級數

We will consider

$$a_1 + a_2 + a_3 + \cdots\cdots$$

e.g.

① $1 + 2 + 3 + \cdots\cdots$ ← divergent

② $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ $\left(= \frac{1}{1-\frac{1}{2}} \right)$
$= 2$

Basic question:

Do we really get a number by an infinite sum? That is, is an (infinite) series convergent?

To make this precise, let us briefly summarize basic facts

on sequences.
数列

Review of sequences

A sequence

$$(a_n)_{n=1}^{\infty} = a_1, a_2, a_3, \cdots, a_n, \cdots$$

is said to be convergent if there exists a number L with the property:

for all there exists set of positive integers

$\forall \varepsilon > 0$, $\exists$ $N = N_\varepsilon$ $(\in \mathbb{N})$ s.t.

in such that

$$|a_n - L| < \varepsilon$$

whenever $n \geq N$

In this case, we say L is the limit of $(a_n)_{n=1}^{\infty}$.

Notation:

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \quad (\text{as } n \to \infty)$$

We say $(a_n)_{n=1}^{\infty}$ is divergent 發散的 if it is not convergent.

Example

Prove that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

pf

$\forall \varepsilon > 0$, $\exists N = \underline{\frac{1}{\varepsilon} + 1}$ s.t.

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N}$$

$$= \frac{1}{\frac{1}{\varepsilon} + 1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

when $n \geq N = \frac{1}{\varepsilon} + 1$ #

To determine the convergence of a seq, we have the following useful results:

(i) Every convergent seq. is bounded.
有界

i.e. $\exists M$ s.t. $|a_n| \le M \ \forall n = 1, 2, \ldots$

Equivalently, every unbounded seq. is divergent.

e.g. $(n)_{n=1}^{\infty} = 1, 2, 3, 4, \ldots$ is divergent

(ii) A bounded above increasing seq. is convergent.
有上界 遞增

Equivently, a bounded below decreasing seq. is convergent.

e.g. $\left(\frac{1}{n+1}\right)_{n=1}^{\infty}$ is decreasing and bounded below by 0, so it is convergent.

(iii) If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent then $(\gamma \in \mathbb{R})$

$$(a_n+b_n)_{n=1}^{\infty}, \quad (\gamma \cdot a_n)_{n=1}^{\infty},$$
$$(a_n \cdot b_n)_{n=1}^{\infty}, \quad (a_n - b_n)_{n=1}^{\infty}$$

are convergent, and

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

....

Furthermore, if $\lim_{n\to\infty} b_n \neq 0$, then $(a_n/b_n)_{n=1}^{\infty}$ is convergent, and

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

$$1$$
$$\|$$
$$1+\left(\lim_{n\to\infty}\frac{1}{n}\right)^2$$
$$\|$$
$$1+\lim_{n\to\infty}\frac{1}{n^2}$$

eg.

$$\lim_{n\to\infty}\frac{(n^2+1)\frac{1}{n^2}}{(n^2+n)\frac{1}{n^2}} = \lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{1+\frac{1}{n}}$$

$$1+0$$
$$=1$$

$$= 1$$

(iv) Assume $\lim_{n\to\infty} a_n = c$ and a function f is continuous at c

Then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

e.g.

$\cos x$ is continuous at $0 = \lim_{n\to\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \cos\frac{1}{n} \overset{\downarrow}{=} \cos\left(\lim_{n\to\infty} \frac{1}{n}\right)$$

$$= \cos 0 = 1$$

(v) Suppose that $\exists\, N$ s.t.

$$a_n \leq b_n \leq c_n \qquad \forall n \geq N$$

and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$$

Then $(b_n)_{n=1}^{\infty}$ is convergent, and

$$\lim_{n\to\infty} b_n = L$$

e.g.

Consider $\left(\frac{\sin(n^2)}{n}\right)_{n=1}^{\infty}$

Since

$$\frac{-1}{n} \leq \frac{\sin(n^2)}{n} \leq \frac{1}{n}$$

and

$$\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$$

we have

$$\lim_{n\to\infty} \frac{\sin(n^2)}{n} = 0$$

(vi) A bounded seq. $(a_n)_{n=1}^{\infty}$ converges

子(數)列

$\Leftrightarrow$ every subsequence of it converges to the same number

$$\Leftrightarrow \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$$

e.g. $a_n = (-1)^n$

$a_{2k} = 1 \ \forall k \to +1$

$a_{2k-1} = -1 \ \forall k \to -1$

$$\overline{\lim_{n\to\infty}} a_n = +1$$

$$\neq$$

$$\lim_{n\to\infty} a_n = -1$$

$\Rightarrow (a_n)_{n=1}^{\infty}$ is divergent.

(vii) A seq of real numbers is convergent iff it is a Cauchy seq.

if "and only if"

A seq. $(a_n)_{n=1}^{\infty}$ is called a Cauchy sequence if

$\forall \varepsilon > 0 \quad \exists N = N_\varepsilon$ s.t.

$$|a_n - a_m| < \varepsilon$$

whenever $n, m \geq N$.

e.g. Assume $|a_{n+1} - a_n| < \left(\frac{1}{2}\right)^n \quad \forall n$

Prove that $(a_n)_{n=1}^{\infty}$ is convergent.

pf

$\forall \varepsilon > 0 \quad \exists N = \underline{(\log_{\frac{1}{2}} \varepsilon) + 10}$ s.t.

$$|a_n - a_m| = |(a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) + a_{n+2} - \cdots - a_{m-1} + a_{m-1} - a_m|$$

($\triangle$ ineq. $|a+b| \leq |a| + |b|$)

$$\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m|$$

$$< \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n+1} + \cdots + \left(\frac{1}{2}\right)^{m-1}$$

$$= \cancel{\frac{\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^m}{1 - \frac{1}{2}}} < \frac{\left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = \left(\frac{1}{2}\right)^{n-1}$$

$$\leq \left(\frac{1}{2}\right)^{N-1} < \left(\frac{1}{2}\right)^{(\log_{\frac{1}{2}} \varepsilon)}$$

$$= \varepsilon$$

$a^{\log_a x} = x$

$\forall \ m > n \geq N$

$\Rightarrow (a_n)_{n=1}^{\infty}$ is Cauchy

$\Rightarrow (a_n)_{n=1}^{\infty}$ is convergent. #

Series 級數

Let $(a_k)_{k=0}^{\infty}$ be a seq.

Consider the seq of partial sums:

Consider the seq. of partial sums:

$$S_n = a_0 + a_1 + \cdots + a_n$$

i.e.,

$$a_0,\ (a_0+a_1),\ (a_0+a_1+a_2),\ \ldots$$

Def

We say that the series $\sum_{k=0}^{\infty} a_k$ converges to L if the seq. $(S_n)_{n=0}^{\infty}$ converges to L:

"sum" of this series

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k\right) = L$$

If $(S_n)_{n=0}^{\infty}$ is divergent, we say the series $\sum_{k=0}^{\infty} a_k$ diverges

Example

$$= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

① $\sum^{\infty} \frac{1}{\ldots} = ?$ $\Big(\stackrel{\text{def}}{=} \ldots$

$$\sum_{k=0} \frac{1}{(k+1)(k+2)} .$$ $\left(= \lim_{n\to\infty} S_n \quad S_n = \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} \right)$

Sol

Since

$$S_n = \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$$

$$= \sum_{k=0}^{n} \left(\frac{1}{k+1} - \frac{1}{k+2} \right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \cdots$$

$$+ \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2} \right)$$

$$= 1 - \frac{1}{n+2} = S_n$$

we have

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+2} \right)$$

$$= 1 \qquad \#$$

② $\sum_{k=0}^{\infty} (-1)^k = ?$

Sol

$$S_n = \sum_{k=0}^{n} (-1)^k = \cancel{1} \cancel{-1} \cancel{+1} \cancel{-1} + 1 - 1 + \cdots (-1)^n$$

$$= \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

Since

$$\lim_{k\to\infty} S_{2k} = 1$$

$$\neq$$

$$\lim_{k\to\infty} S_{2k-1} = 0$$

$\Rightarrow$ $(S_n)_{n=1}^{\infty}$ has two subseq with different limits

the seq $(S_n)_{n=1}^{\infty}$ diverges.

$\Rightarrow \sum_{k=0}^{\infty} (-1)^k$ diverges #