Geometric models for noncommutative algebras
Fall 2025, week 9

<u>Recall</u>

Let M be a smooth mfd.

A Poinsson str on M is a bilinear map

$$\{ , \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

s.t.

- $\{f, g\} = -\{g, f\}$
- $\{f, gh\} = \{f, g\}h + g\{f, h\}$
- Jacobi.

Such a bracket $\{ , \}$ is equivalenet to a bivector field   Poisson bivector

$$\pi \in \Gamma(\Lambda^2 T_M)$$

with the property

$$[\pi, \pi] = 0$$

The correspondence is given by $f \circ \phi$ $\xrightarrow{\phi}\ f \to \mathbb{R}$

$$\{f, g\} = \langle \pi, df \wedge dg \rangle$$

$\{f,g\} = \langle \Pi, df\wedge dg\rangle$ $\quad M \xrightarrow{\phi} N$

Let $(M, \overset{\Pi_M}{\underline{\{\,,\}_M}})$, $(N, \overset{\Pi_N}{\underline{\{\,,\}_N}})$ be Poisson mfds.

Def

$\phi^*: C^\infty(N) \to C^\infty(M)$, $f \mapsto f\circ\phi$

A smooth map $\phi: M\to N$ is a Poisson map if

$$\{\phi^*f, \phi^*g\}_M = \phi^*\{f,g\}_N \qquad \forall f,g\in C^\infty(N)$$

Lemma

ϕ is a Poisson map

$\iff \quad \phi_*(\Pi_M|_p) = \Pi_N|_{\phi(p)} \qquad \forall p\in M$

pf

ϕ is a Poisson map

$\iff$

$$\{\phi^*f, \phi^*g\}_M = \phi^*\{f,g\}_N = \phi^*\langle \Pi_N, df\wedge dg\rangle$$

$$\| $$

$$\langle \Pi_M, \underbrace{d(\phi^*f)\wedge d(\phi^*g)}_{\phi^*(df\wedge dg)} \rangle$$

$$\|$$

$$\langle \Pi_M, \phi^*(df\wedge dg)\rangle$$

$\Pi_M \in \Gamma(\Lambda^2 TM)$

$\| $

$\{\sum_i V_i\wedge W_i \mid V_i, W_i \in \Gamma(TM)\}$

$\sum V_i\wedge W_i$

$$(\phi^*(df\wedge dg))_p(V_p, \overset{T_pM}{W_p}) = (df\wedge dg)_{\phi(p)}(\phi_*V_p, \phi_*W_p)$$

if $\Pi_M = \sum v_i \wedge w_i$, then

$= \quad p \longmapsto \sum_i \left(\Phi^*(df \wedge dg)\right)_p (v_i|_p, w_i|_p)$

$\qquad\qquad\qquad\qquad \shortparallel$

$(df \wedge dg)_{\Phi(p)} \left(\Phi_*(v_i|_p), \Phi_*(w_i|_p)\right)$

So

$$\langle \Pi_M, \Phi^*(df \wedge dg) \rangle (p)$$

$$= \sum_i \langle \Phi_*(v_i|_p) \wedge \Phi_*(w_i|_p), (df \wedge dg)_{\Phi(p)} \rangle$$

$$= \langle \Phi_*(\Pi_{M_p}), (df \wedge dg)_{\Phi(p)} \rangle$$

$$= \left(\Phi^* \langle \Pi_N, df \wedge dg \rangle\right)(p)$$

$$= \langle \Pi_N|_{\Phi(p)}, (df \wedge dg)_{\Phi(p)} \rangle$$

So

$$\Phi_*(\Pi_M|_p) = \Pi_N|_{\Phi(p)} \quad \forall\, p \in M. \quad \#$$

<u>Def</u>

A submfd S of a Poisson mfd (M, Π)

is a Poisson submanifold if

(i) S is a Poisson mfd

(ii) the inclusion $S \overset{i}{\hookrightarrow} M$ is a Poisson map

Note that if S is a Poisso submfd of (M, Π) then its Poisson str. is determined by Π, since

$$i_*(\Pi_S|_p) = \Pi_M|_{i(p)} = \Pi_M|_p$$

and $i_* : \Gamma(\Lambda^2 T_pS) \to \Gamma(\Lambda^2 T_pM)$ is 1-1

Remark

Let S be a Poisson submfd.

$\Leftrightarrow$ $i^* : C^\infty(M) \to C^\infty(S)$
$\{i^*\tilde{f}, i^*\tilde{g}\} = i^*\{\tilde{f}, \tilde{g}\}$

$\tilde{g}, \tilde{f} : M \to \mathbb{R}$
$\tilde{f}|_S = f$, $\tilde{g}|_S = g$

$\forall f, g \in C^\infty(S)$, $\forall$ extensions $\tilde{f}, \tilde{g} \in C^\infty(M)$ of f, g

we have

$$\{\underset{i^*\tilde{f}}{f}, \underset{i^*\tilde{g}}{g}\}_S = i^*\{\tilde{f}, \tilde{g}\} = \{\tilde{f}, \tilde{g}\}|_S$$

In particular,

$$\{\tilde{f}, \tilde{g}\}|_S = \{\tilde{\tilde{f}}, \tilde{\tilde{g}}\}|_S$$

if $\tilde{f}|_S = \tilde{\tilde{f}}|_S = f$, $\tilde{g}|_S = \tilde{\tilde{g}}|_S = g$

<u>Prop</u>

Let (M, Π) be a Poisson mfd, S a submfd.

Then

(i) $\Pi^\#(T^*M)|_S \subseteq TS$

Recall: $\Pi^\# : T^*M \to TM$

$\alpha \longmapsto \iota_\alpha(\Pi)$

$\Updownarrow$

(ii) Every Hamiltonian vector field X_f, $f \in C^\infty(M)$, is tangent to S.

$\Updownarrow$

(iii) S is a Poisso submfd. $\Gamma(TM)$

$(\iota_{df}(\Pi))(g) = \langle \iota_{df}(\Pi), dg \rangle = \langle \Pi, df \wedge dg \rangle$

$= \{f, g\}$

$= X_f(g)$

pf

(i) $\Leftrightarrow$ (ii) because $X_f = \Pi^\#(df)$

(iii) $\Rightarrow$ (i):

Assume S is a Poisson submfd., $i : S \hookrightarrow M$

$\Rightarrow \quad i_*(\Pi_S|_p) = \Pi|_p \quad \forall p \in S$

$\Rightarrow \forall \alpha \in T^*M$,

$$\Pi^\#(\alpha)|_p = \iota_\alpha \Pi_p = \iota_\alpha(\underbrace{i_*(\Pi_S|_p)}_{\parallel})$$

$$= i_\alpha \left(\sum_j i_* V_j|_p \wedge i_* W_j|_p \right)$$

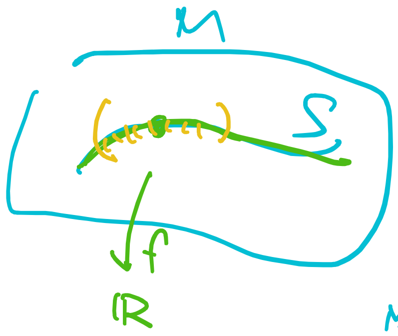
$$= \sum_j \underbrace{\overset{\mathbb{R}}{i_\alpha (i_*(V_j|_p))}} \cdot i_*(W_j|_p) \pm i_*(V_j|_p) \cdot \underbrace{\overset{\overset{\mathbb{R}}{0}}{i_\alpha (i_* W_j|_p)}}$$

$$\in \operatorname{im}(i_*) = TS$$

$\mathbb{R}^2 = M$

$S = x\text{-axis}$ $\qquad f(x)$

$\Rightarrow$ (i)



(ii) $\Rightarrow$ (iii): Assume (ii) holds.

Given any $f, g \in C^\infty(S)$, $p \in S$, define

$$\circledast \quad \{f, g\}_S(p) = \{\tilde f, \tilde g\}(p)$$

where $\tilde f, \tilde g \in C^\infty(M)$ s.t. $\exists_{p \in} U \underset{\text{open}}{\subset} S$,

$$\tilde f|_U = f|_U, \quad \tilde g|_U = g|_U$$

$M = \mathbb{R}$

$S = \mathbb{R} \setminus \{0\}$

$f = \frac{1}{x}$

It suffices to show $\circledast$ is independent of the choice of $\tilde f$ and $\tilde g$.

If $\tilde{\tilde g}$ is another such function. ($\tilde{\tilde g} \in C^\infty(M)$, $\tilde{\tilde g}|_U = g$) then

$$\{\tilde f, \tilde g\}(p) - \{\tilde f, \tilde{\tilde g}\}(p)$$

$$= \{\tilde f, \tilde g - \tilde{\tilde g}\}(p) = (X_{\tilde f} \lrcorner (\tilde\alpha - \tilde{\tilde\alpha})$$

$$= 0$$

$= 0$ on $U_p \underset{\text{open}}{\subset} S$

That is,

$$\{\tilde{f}, \tilde{g}\}(p) = \{\hat{\tilde{f}}, \hat{\tilde{g}}\}(p)$$

\#

Local Structure of Poisson manifolds

Let (M, Π) be a Poisson mfd.

Def

Choose a local coord sys. around a point $p \in M$. $n = \dim M$.

$$\Rightarrow \quad \Pi = \frac{1}{2}\sum_{i,j=1}^{n} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

The rank of Π at p

$$= \operatorname{rank}\left(\Pi_{ij}(p)\right)_{n\times n}$$

exer: independent of choice of coord.

Remark

① $(\Pi_{ij}(p))$ is skew-symmetric

$\Rightarrow$ rank$(\Pi_{ij}(p))$ is even

② rank $\in \mathbb{Z}_{\geq 0}$, not necessarily constant.

③ M is symplectic   matrix repn $= (-1)(\Pi_{ij}(p))$

$\Leftrightarrow$ $\Pi^\# : T_p^*M \to T_pM$ is an iso $\forall p \in M$.

$\Leftrightarrow$ $\Pi|_p$ has maximal rank $\forall p \in M$.

<u>Cor</u> (① + ③)

dim(symplectic mfd) is even

<u>Thm</u> (Weinstein's splitting thm)

Let (M, Π) be an n-dimensional Poisson mfd, $x_0 \in M$, $k \in \mathbb{Z}$ s.t. rank$(\Pi|_{x_0}) = 2k$.

Then $\exists$ local coord. sys

$$(q_1, \cdots, q_k, p_1, \cdots, p_k, y_1, \cdots, y_l)$$

centered at x_0 with $l = n - 2k$. such

$\{q_i, p_j\} = \delta_{ij}$ $\{q_i, q_j\} = 0$ $\{p_i, p_j\} = 0$

$\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = 0$, $\{p_i, p_j\} = 0$

$$\{y_i, y_j\} = \Phi_{ij}(y_1, \dots, y_\ell), \qquad \Phi_{ij}(\underbrace{0, \dots, 0}_{\updownarrow \; x_0}) = 0$$

$$\{y_i, q_j\} = 0 \qquad \{y_i, p_j\} = 0$$

In this coord. sys.

$$\Pi_{ij} = \begin{array}{c} \\ q \\ p \\ y \end{array} \begin{array}{c} \begin{array}{ccc} q & p & y \end{array} \\ \begin{pmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & \Phi_{ij} \end{pmatrix} \end{array}$$

<u>Remark</u>

If $\mathbb{R}^n$ is equipped with a Poisson str.,
then it decomposes as

$$\mathbb{R}^n = \mathbb{R}^{2k} \times \mathbb{R}^\ell$$

where $\mathbb{R}^{2k}$ is symplectic,
$\mathbb{R}^\ell$ carries a Poisson str. vanishing at $(0, 0, \dots, 0)$

To prove the thm, recall the ...

... the Flow-Box thm and
the Frobenius thm:

Thm (Flow-Box Thm).

Let $X \in \mathfrak{X}(M)$, $p \in M$ s.t. $X|_p \neq 0$.

Then $\exists$ local coord. $(x_1, \dots, x_n)$ on $U \ni p$ ($U \underset{\text{open}}{\subset} M$)
s.t.

$$X = \frac{\partial}{\partial x_1}$$

on U.

Let M be a mfd. Recall that

$$S = \coprod_{x \in M} S_x,$$

s.t. S_x is a vector subsp of $T_x M$ $\forall x \in M$, is called a <u>distribution</u>.

A distribution S is smooth if $\forall p \in M$ $\exists \underbrace{X_1, \dots, X_k}$ (smooth vector fields on a nbd of p, U)
s.t.

$$S_x = \operatorname{span}\{X_1|_x, \dots, X_k|_x\}$$

$\forall x \in U$

In the sequel, we shall only consider smooth distributions.

The rank of S at $x \in M$ $=$ $\dim S_x$

Thm (Frobenius Thm, local form)
Let S be a smooth distribution of constant rank r on an n-dimensional mfd M.
The following are equivalent:

(i) S is involutive, i.e. $\forall X, Y \in \mathfrak{X}(M)$ with

$$X_p, Y_p \in S_p \quad \forall p \in M,$$

we have

$$[X, Y]_p \in S_p \quad \forall p \in M$$

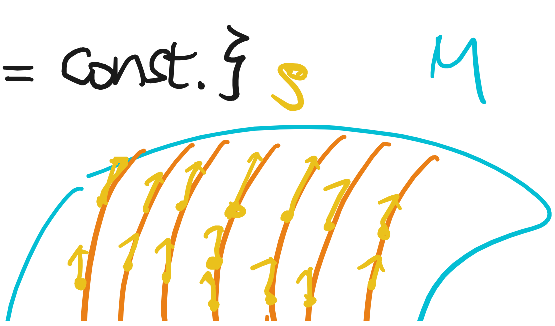
(ii) $\forall p \in M$, $\exists$ local chart $(U; x_1, \cdots, x_n)$ centered at p s.t.

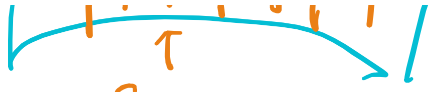
$$S|_U = \operatorname{span}\left\{ \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_r} \right\}$$

In this case, S is called integrable, and

$$\to \{ x_{r+1} = \text{const}, \quad \cdots, \quad x_n = \text{const}. \}$$

are the leaves of S





$\{x_{r+1}=\text{Const}, \dots, x_n=\text{Const}\}$

pf of Weinstein's Thm (Induction on $\underbrace{\operatorname{rank}(\Pi|_{x_0})}_{=\rho}$)

If $\rho=0$, then $\Pi|_{x_0}=0$, $\Updownarrow$ $\Pi_{ij}(x_0)=0$

Set $\phi_{ij}=\Pi_{ij}$

Assume $\rho>0$.

$\Rightarrow \exists f,g\in C^\infty(M)$ s.t. $\{f,g\}(x_0)\neq 0$

$\Leftrightarrow X_g(f)(x_0)\neq 0$

Set

$$p_1(x)=g(x)-g(x_0)$$

By the flow-box thm, $\exists$ coord. $q_1,\dots$ around x_0 s.t.

$$X_g = X_{p_1} = \frac{\partial}{\partial q_1}$$

$$\Rightarrow \{p_1,q_1\}=1=X_{p_1}(q_1)$$

Also, since

$$\begin{pmatrix} \underbrace{X_{p_1}(p_1)}_{=\{p_1,p_1\}=0} & X_{q_1}(p_1) \\ X_{p_1}(q_1) & X_{q_1}(q_1) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is invertible, [1] the vector fields X_{p_1} and X_{q_1} are linearly independent.

Furthermore,

$$[X_{q_1}, X_{p_1}] = X_{\{q_1, p_1\}} = X_{-1} = 0$$

$\Rightarrow$ X_{q_1} and X_{p_1} generate a rank-two integrable distribution a nbd of x_0.

By the Frobenius thm, $\exists$ local coord. $y_3, \cdots, y_n$ s.t. X_{q_1} and X_{p_1} are tangent to

$$\{y_3 = \text{const}, \cdots, y_n = \text{const}\}.$$

$\Rightarrow$

$$\{q_1, y_i\} = X_{q_1}(y_i) = 0 \qquad \forall i = 3, \cdots, n.$$

$$\{p_1, y_i\} = X_{p_1}(y_i) = 0$$

Since $X_{q_1}(q_1) = X_{q_1}(y_i) = 0$, $X_{q_1}(p_1) = -1$,

$$X_{q_1} = -\frac{\partial}{\partial p_1}$$

Want:

$$\Pi_{ij} = \begin{array}{c|ccc} & q_1 & p_1 & y \\ \hline q_1 & 0 & 1 & 0 \\ p_1 & -1 & 0 & \\ y & 0 & & \phi_{ij}(y) \end{array}$$

Claim $\{y_i, y_j\}$ are indep. of q_1, p_1

By the Jacobi id,

$$\underbrace{\boxed{\{\{y_i, y_j\}, q_1\}}}_{=0} + \{\cancel{\{y_j, q_1\}}^{=0}, y_i\} + \{\cancel{\{q_1, y_i\}}^{=0}, y_j\} = 0$$

$$\Rightarrow X_{q_1}(\{y_i, y_j\}) = 0$$

$$\parallel$$

$$-\frac{\partial}{\partial p_1}(\{y_i, y_j\})$$

Similarly, $\frac{\partial}{\partial q_1}(\{y_i, y_j\}) = 0$

$\Rightarrow$ $\{y_i, y_j\}$ depend only $(y_3, \dots, y_n)$

Then apply the induction hypothesis to $\uparrow$

$\Rightarrow$ ok. #

Let $x_0 \in M$. $\quad \Pi|_U = \left(\begin{array}{cc|c} 0 & I & 0 \\ -I & 0 & \\ \hline 0 & & \phi_{ij} \end{array}\right)$

$\exists U$, $x_0 \in U \underset{\text{open}}{\subset} M$

symplectic

$\phi_{ij}|_{x_0} = 0$

Consider (M,π), $\pi|_{x_0}=0$

Assume $y=(y_1,\cdots,y_n)$ is a local chart, centered at x_0 s.t.

$$\{y_i,y_j\}=\phi_{ij}(y) \quad \text{with} \quad \phi_{ij}(0)=0.$$

Consider the Taylor expansions of ϕ_{ij} around 0:

$$\phi_{ij}(y)=\underbrace{\cancel{\phi_{ij}(0)}}_{=0}+\boxed{\sum_{k=1}^{n}\underbrace{\boxed{\frac{\partial\phi_{ij}}{\partial y_k}(0)}}_{C_{ij}^{k}}\cdot y_k}+O(y^2)$$

By skew-symmetry and Jacobi, one can show

$$C_{ij}^{k}=-C_{ji}^{k}$$

$$\sum_{l=1}^{n}\left(C_{ij}^{l}C_{lk}\cdots-\quad\right)=0$$

$\Leftrightarrow$ C_{ij}^{k} are the structure constants of a Lie algebra.