Geometric models for noncommutative algebras
Fall 2025, week 7

<u>Recall</u>

$$[\ ,\ ] : \Gamma(\Lambda^a T_M) \times \Gamma(\Lambda^b T_M) \to \Gamma(\Lambda^{a+b-1} T_M)$$

- $[A,B] = -(-1)^{(a-1)(b-1)}[B,A]$, $\quad A \in \Gamma(\Lambda^a T_M)$, $B \in \Gamma(\Lambda^b T_M)$

- $[A, B\wedge C] = [A,B]\wedge C + (-1)^{(a-1)b} B \wedge [A,C]$, $\quad C \in \Gamma(\Lambda^c T_M)$

- $$(-1)^{(a-1)(c-1)}[A,[B,C]] + (-1)^{(b-1)(a-1)}[B,[C,A]] + (-1)^{(c-1)(b-1)}[C,[A,B]] = 0$$

which extend the following relations

$$[f,g] = 0 \qquad \forall f,g \in C^\infty(M)$$

$$[X,f] = X(f) \qquad \forall X \in \mathfrak{X}(M)$$

$$[X,Y] = [X,Y]_{Lie} \qquad Y \in \mathfrak{X}(M)$$

<u>Remark</u>

$(\mathbb{Z}_-)$

$\mathfrak{g}$ is a graded Lie algebra if

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i \qquad \left(\mathfrak{g}_i = \Gamma(\Lambda^{i+1} T_M)\right)$$

and

$$[X,Y] = -(-1)^{|X||Y|}[Y,X]$$

$$[X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]]$$

Remark

An element in $\Gamma(\Lambda^k T_M)$ induces a multilinear map

$$\underbrace{C^\infty(M)\times\cdots\times C^\infty(M)}_{k \text{ times}} \longrightarrow C^\infty(M)$$

$(X_i \in \Gamma(T_M))$

by

NOTE: $X\wedge Y = -Y\wedge X$

$$(X_1\wedge\cdots\wedge X_k)(\overset{f_1\cdot g_1}{f_1},\dots,f_k)$$

sign of σ

$$= \sum_{\sigma\in S_k} (-1)^\sigma X_{\sigma(1)}(f_1)\cdot X_{\sigma(2)}(f_2)\cdots X_{\sigma(k)}(f_k)$$

$\sigma: \{1,\dots,k\} \xrightarrow[\text{onto}]{1-1} \{1,\dots,k\}$

Consider $M=\mathbb{R}^n$, and

$$\{f,g\} = \sum^n \Pi_{ij} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

If we set

$$\pi := \frac{1}{2} \sum_{i,j=1}^{n} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \in \Gamma(\Lambda^2 T_{\mathbb{R}^n})$$

then by Remark, we have an induced bilinear map

$$\pi : C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

given by the formula.

$$\pi(f,g) = \frac{1}{2} \sum_{i,j=1}^{n} \pi_{ij} \underbrace{\left(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\right)(f,g)}_{\begin{gathered} \parallel \\ \frac{\partial}{\partial x_i}(f) \cdot \frac{\partial}{\partial x_j}(g) \\ - \frac{\partial}{\partial x_j}(g) \frac{\partial}{\partial x_i}(f) \end{gathered}}$$

$$\{x_i, x_j\} = -\{x_j, x_i\} = -\pi_{ji}$$

$$= \frac{1}{2}\Big(\sum_{i,j} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \underbrace{- \sum_{i,j} \pi_{ij} \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j}}_{+ \sum_{i,j} \pi_{ji} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i}} \Big)$$

$$= \sum^{n} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} = \{f, g\}$$

$$\sum_{i,j=1} {}_{ij} \, \partial x_i \, \partial x_j = \{i, j\}$$

So

$$\pi = \frac{1}{2} \sum_{i,j=1}^{n} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

is another way to express a Poisson str. on $\mathbb{R}^n$

<u>Lemma</u>

If

$$\pi = \frac{1}{2} \sum_{i,j=1}^{n} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$ (not necessarily from a Possion str.)

and $f \in C^\infty(\mathbb{R}^n)$.

then

$$[\pi, f] = \text{"}X_f\text{"} = \overset{?}{(-1)} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \pi_{ji} \right) \cdot \frac{\partial}{\partial x_i}$$

$$[\pi, \pi]$$

$$= \overset{?}{+2} \sum_{1 \le i < j < k \le n} \left(\sum_{\ell=1}^{n} \left(\pi_{k\ell} \frac{\partial \pi_{ij}}{\partial x_\ell} + \pi_{i\ell} \frac{\partial \pi_{jk}}{\partial x_\ell} + \pi_{j\ell} \frac{\partial \pi_{ki}}{\partial x_\ell} \right) \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$$

Pf

$\Gamma(\wedge^2 T\mathbb{R}^n)$ $\in$

+1 ||

$a=0$ $b=1$ $c=1$

A $B \wedge C$

① $[\pi, f] = -(-1)^{1\cdot(-1)} [f, \pi] = [f, \frac{1}{2}(\pi_{..} \frac{\partial}{\cdot}) \wedge \frac{\partial}{\cdot}]$

$$= \frac{1}{2}\Big([f, \Pi_{ij}\frac{\partial}{\partial x_i}] \wedge \frac{\partial}{\partial x_j} + (-1)^{(0-1)\cdot 1}\Pi_{ij}\frac{\partial}{\partial x_i}\wedge [f, \frac{\partial}{\partial x_j}] \Big)$$

$a=0 \quad b=0 \quad c=1$

$A \quad B \wedge C$

$= -1$

$[f, \Pi_{ij}]\wedge \frac{\partial}{\partial x_i} + (-1)^{(0-1)\cdot 0}\,\Pi_{ij}\wedge[f, \frac{\partial}{\partial x_i}]$

$-[\frac{\partial}{\partial x_i}, f] = -\frac{\partial f}{\partial x_i}$

$-\frac{\partial f}{\partial x_j}$

Π_{ji}

$$= \frac{1}{2}\Big(\sum_{i,j} -\Pi_{ij}\frac{\partial f}{\partial x_i}\cdot\frac{\partial}{\partial x_j} + \sum_{i,j}\Pi_{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}\Big)$$

$= \sum_{i,j}\Pi_{ji}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}$

$$= \sum_{j=1}^{n}\Big(\sum_{i=1}^{n}\Pi_{ji}\frac{\partial f}{\partial x_i}\Big)\frac{\partial}{\partial x_j}$$

$$= -X_f$$

Recall

$$\Big(X_f = \{f, -\} = \sum \Pi_{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} \Big)$$

② $[\Pi, \Pi] = \frac{1}{2}[\Pi, \Pi_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}]$

$$= \frac{1}{2}\Big([\Pi, \Pi_{ij}]\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} + \Pi_{ij}[\Pi, \frac{\partial}{\partial x_i}]\wedge\frac{\partial}{\partial x_j} + \Pi_{ij}\frac{\partial}{\partial x_i}\wedge[\Pi, \frac{\partial}{\partial x_j}]\Big)$$

$(-1)^{(2-1)\cdot 0} = +1 \qquad (-1)^{(2-1)\cdot 1} = -1$

$[\Pi, \Pi_{ij}] = -X_{\Pi_{ij}}$

$[\Pi, \frac{\partial}{\partial x_i}] = -(-1)^{(1-1)(2-1)}[\frac{\partial}{\partial x_i}, \Pi]$

$= -\frac{1}{2}[\frac{\partial}{\partial x_i}, \Pi_{kl}\frac{\partial}{\partial x_k}\wedge\frac{\partial}{\partial x_l}]$

$\frac{\partial \Pi_{kl}}{\partial x_i}$

$= -\frac{1}{2}\Big([\frac{\partial}{\partial x_i}, \Pi_{kl}]\frac{\partial}{\partial x_k}\wedge\frac{\partial}{\partial x_l}$

$\pm \Pi_{kl}\cancel{[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}]}\wedge\frac{\partial}{\partial x_l} = 0$

$\pm \Pi_{kl}\frac{\partial}{\partial x_k}\wedge\cancel{[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_l}]}\Big) = 0$

$-\Pi_{kl}\frac{\partial \Pi_{ij}}{\partial x_k}\frac{\partial}{\partial x_l}$

$\|$

$$= \frac{1}{2}\Big(-X_{\Pi_{ij}} \wedge \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \Pi_{ij}\left(-\frac{1}{2}\right) \frac{\partial \Pi_{kl}}{\partial x_i} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l} \wedge \frac{\partial}{\partial x_j}$$

$$- \underbrace{\Pi_{ij} \frac{\partial}{\partial x_i} \wedge \left(-\frac{1}{2} \frac{\partial \Pi_{kl}}{\partial x_j} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l} \right)} \Big)$$

$$= \frac{1}{2} \Pi_{ij} \frac{\partial \Pi_{kl}}{\partial x_j} \left(\frac{\partial}{\partial x_i} \wedge \underline{\frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l}} \right)$$

$$= \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l} \wedge \frac{\partial}{\partial x_i}$$

$$-\frac{1}{2}\Pi_{ij}$$

$$\shortparallel$$

$$= \frac{1}{2} \Pi_{ji} \frac{\partial \Pi_{kl}}{\partial x_i} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l} \wedge \frac{\partial}{\partial x_j}$$

$$= -\frac{1}{2}\left(\Pi_{kl} \frac{\partial \Pi_{ij}}{\partial x_k} \underbrace{\frac{\partial}{\partial x_l} \wedge \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}}_{= \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_l}} + \Pi_{ij} \frac{\partial \Pi_{kl}}{\partial x_i} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l} \wedge \frac{\partial}{\partial x_j} \right)$$

$k \; l \quad k \quad j \quad l$

$k \to i$
$l \to j$
$j \to l$
$i \to k$

$$= \sum_{i,j,k=1}^{n} \left(\sum_{l=1}^{n} \Pi_{kl} \frac{\partial \Pi_{ij}}{\partial x_l} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$$

$k \leftrightarrow l$

$$= \sum_{1 \le i < j < k \le n} \Big(\qquad \Big) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$$

#

Poisson bivector fields

Recall we have

$$\langle - | - \rangle : \Gamma(T_M) \times \Gamma(T_M^*) \to C^\infty(M),$$

$$\langle X | \alpha \rangle := \alpha(X)$$

$\forall\, \alpha \in \Gamma(T_M^*)$, $X \in \Gamma(T_M)$.

$\rightsquigarrow$

$$\langle - | - \rangle : \Gamma(\Lambda^\bullet T_M) \times \Gamma(\Lambda^\bullet T_M^*) \to C^\infty(M)$$

$$\langle X_1 \wedge \cdots \wedge X_k \,|\, \alpha_1 \wedge \cdots \wedge \alpha_\ell \rangle$$

$$:= \begin{cases} \sum_{\sigma \in S_k} (-1)^\sigma \langle X_{\sigma(1)} | \alpha_1 \rangle \cdots\cdots \langle X_{\sigma(k)} | \alpha_k \rangle, & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

<u>Lemma</u>

An $\mathbb{R}$-multilinear map

$$\Phi : \overbrace{C^\infty(M) \times \cdots \times C^\infty(M)}^{k \text{ times}} \to C^\infty(M)$$

is <u>skew-symmetric</u> (i.e. $\Phi(\cdots f, g, \cdots) = -\Phi(\cdots g, f, \cdots)$)

and satisfies the Leibniz rule i.e.

$$\Phi(fg, f_2, f_3, \dots, f_k) = \Phi(f, f_2, \dots, f_k) \cdot g + f \cdot \Phi(g, f_2, \dots, f_k)$$

$\Leftrightarrow$ $\exists\ \Xi \in \Gamma(\Lambda^k TM)$ s.t.

$$\Phi(f_1, \dots, f_k) = \langle \Xi \mid df_1 \wedge \dots \wedge df_k \rangle$$

$$\forall f_1, \dots, f_k \in C^\infty(M)$$

pf: Dufour and Nguyen, Poisson str and their normal forms

Lemma 1.2.2.

<u>Prop</u>

A manifold $(M, \{\ ,\ \})$ is a Poisson mfd.

$\Leftrightarrow$ $\exists\ \pi \in \Gamma(\Lambda^2 TM)$ s.t. $[\pi, \pi] = 0$

"Poisson bivector (field)" "Poisson tensor"

In this case,

$$\{f, g\} = \pi(f, g) = \langle \pi \mid df \wedge dg \rangle \quad (*)$$

pf

"$\Rightarrow$"

Recall $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$

satisfies (i) skew-symm. (ii) Leibniz (iii) Jacobi.

By Lemma, (i) + (ii) $\Leftrightarrow$ $\exists\, \Pi \in \Gamma(\Lambda^2 T_M)$ s.t. $\circledast$ holds

In local coordinates,

$\frac{1}{2}\sum_{i,j} \Pi_{ij} \frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$ ''

$df = \sum_k \frac{\partial f}{\partial x_k} dx_k$

$$\{f,g\} = \langle \Pi \mid df\wedge dg\rangle$$

$dg = \sum_\ell \frac{\partial g}{\partial x_\ell} dx_\ell$

$$\langle \tfrac{\partial}{\partial x_i}\wedge\tfrac{\partial}{\partial x_j} \mid dx_k\wedge dx_\ell\rangle = \underbrace{\langle \tfrac{\partial}{\partial x_i} \mid dx_k\rangle}_{\delta_{ik}} \underbrace{\langle \tfrac{\partial}{\partial x_j} \mid dx_\ell\rangle}_{\delta_{j\ell}} - \underbrace{\langle \tfrac{\partial}{\partial x_j} \mid dx_k\rangle}_{\delta_{jk}} \underbrace{\langle \tfrac{\partial}{\partial x_i} \mid dx_\ell\rangle}_{\delta_{i\ell}}$$

$$= \left\langle \frac{1}{2}\Pi_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} \,\middle|\, \left(\frac{\partial f}{\partial x_k}dx_k\right)\wedge\left(\frac{\partial g}{\partial x_\ell}dx_\ell\right)\right\rangle$$

$$\overset{\searrow}{=} \sum_{i,j=1}^{n} \Pi_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} = \text{the Formula on } \mathbb{R}^n$$

$$\overset{\text{Lemma}}{\Rightarrow} \quad [\Pi,\Pi] = 0$$

"$\Leftarrow$" is similar #

By Lemma,

$$X_f = \{f,-\} = \overset{\pm ?}{-}[\Pi, f].$$

<u>Cor</u>

Let (M,Π) be a Poisson mfd,

$$L_{X_f}\Pi = 0$$

i.e. Π is preserved by Hamiltonian flows

Π is preserved by Hamiltonian flows,

pf

$$L_{X_f}\Pi = [X_f,\Pi] = -[\underbrace{[\Pi,f]}_{\Gamma(\Lambda^1 T_M)},\underbrace{\Pi}_{\in\Gamma(\Lambda^2 T_M)}]$$

$$= \cancel{-}\,[\Pi,[\Pi,f]] \qquad -(-1)^{(1-1)(2-1)} \quad = -1$$

$$= [[\Pi,\Pi],f] + (-1)^{(2-1)(2-1)}[\Pi,[\Pi,f]] \qquad = -1 \qquad -(-1)^{(2-1)(1-1)}$$

$$= +[[\Pi,f],\Pi]$$

$$= -L_{X_f}\Pi$$

$$\Rightarrow \quad L_{X_f}\Pi = 0 \qquad \#$$

Symplectic manifolds as Poisson mflds

Let Π be a Poisson bivector on M, $p\in M$.

Consider

$$\Pi_p^{\#} : T_p^*M \to T_pM,$$
$$\alpha \longmapsto \iota_\alpha \Pi$$

where $\iota_\alpha\Pi \in T_pM$ is characterized by

$$\langle \iota_\alpha \Pi \mid \beta \rangle = \langle \Pi \mid \alpha \wedge \beta \rangle$$

$\forall \beta \in T_p^* M$.

Remark

The contraction operator $\iota_\alpha : \Gamma(\wedge T_M) \to \Gamma(\wedge T_M)$

is a derivation of degree -1, i.e.

$$\iota_\alpha : \Gamma(\wedge^k T_M) \to \Gamma(\wedge^{k-1} T_M)$$

and

$$\iota_\alpha(\xi \wedge \eta) = \iota_\alpha(\xi) \wedge \eta + (-1)^{\ell} \xi \wedge \iota_\alpha(\eta)$$

$$\forall \xi \in \Gamma(\wedge^\ell T_M),\ \eta \in \Gamma(\wedge T_M)$$

s.t.

$$\iota_\alpha(X) = \alpha(X) \quad \forall X \in \Gamma(T_M)$$

$$\iota_\alpha(f) = 0 \quad \forall f \in C^\infty(M).$$

Lemma

Let $M = \mathbb{R}^n$. If

$$\Pi = \frac{1}{2} \sum_{i,j=1}^{n} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

then the standard matrix representation of

$$\Pi_p^\# : T_p^* \mathbb{R}^n \longrightarrow T_p \mathbb{R}$$

$$\begin{array}{ccc} \cong & & \cong \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

is $\quad -\Pi_{ij}(p)$.

pf

Compute matrix repn w.r.t. dx_k and $\frac{\partial}{\partial x_\ell}$:

$$\Pi_p^{\#}(dx_k) = \iota_{dx_k}\left(\frac{1}{2}\sum_{i,j=1}^{n}\Pi_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}\right)$$

$$= \frac{1}{2}\sum_{i,j=1}^{n}\Pi_{ij}\Bigg(\underbrace{\iota_{dx_k}\left(\frac{\partial}{\partial x_i}\right)}_{\delta_{ki}}\wedge\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_i}\wedge\underbrace{\left(\iota_{dx_k}\frac{\partial}{\partial x_j}\right)}_{\delta_{kj}}\Bigg)$$

$$= \frac{1}{2}\left(\underbrace{\sum_{j=1}^{n}\Pi_{kj}\frac{\partial}{\partial x_j}}_{j\to\ell} - \underbrace{\sum_{i=1}^{n}\Pi_{ik}\frac{\partial}{\partial x_i}}_{i\to\ell}\right)$$

$$= -\sum_{\ell=1}^{n}\Pi_{\ell k}\frac{\partial}{\partial x_\ell}$$

$$\Rightarrow \text{ matrix repn} = -\underbrace{\begin{pmatrix} \Pi_{11} & \cdots & \Pi_{1n} \\ \Pi_{21} & \ddots & \vdots \\ \vdots & & \\ \Pi_{n1} & \cdots & \Pi_{nn} \end{pmatrix}}_{=(\Pi_{ij})}$$

$$= -\left(\Pi_{ij}(p)\right)$$

$\square$

If Π is non-degenerate, i.e.

$$\Pi_p^\# : T_p^*M \to T_pM \text{ is iso } \forall p \in M,$$

then we have

$$\omega_p^\# := (\Pi_p^\#)^{-1} : T_pM \to T_p^*M.$$

which defines a bilinear map

$$\omega_p : T_pM \times T_pM \to \mathbb{R},$$

$$(*) \quad \omega_p(X, Y) := \left(\omega_p^\#(X)\right)(Y) = \left((\Pi_p^\#)^{-1}(X)\right)(Y)$$

for $X, Y \in T_pM$.

Lemma

If

$$(*) \quad \left[\omega_p^\#\right]_{(dx_j)}^{\left(\frac{\partial}{\partial x_i}\right)} = (\omega_{ij})^T = (\omega_{ji})$$

then

$$\omega_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \omega_{ij}.$$

pf

$\circledast \Leftrightarrow (\omega_p^{\#})\left(\frac{\partial}{\partial x_i}\right) = \sum_{k=1} \omega_{ik}\, dx_k$

$\Rightarrow \omega_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left(\sum_{k=1}^{n} \omega_{ik}\, dx_k\right)\left(\frac{\partial}{\partial x_j}\right)$

$= \omega_{ij}.$ #

Prop

If Π is non-degenerate, then $\circledast$ defines a symplectic form (i.e. nondegenerate closed 2-form) on M.

pf

Let $(x_1, \dots, x_n)$ be an arbitrary local coordinate sys on M.

Assume

$$\Pi = \frac{1}{2} \sum_{i,j=1}^{n} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

By the assumption, (Π_{ij}) is invertible

Claim:

$$\omega_{ij} = \omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = (\Pi_{ij})^{-1}$$

satisfies

① $\omega_{ij} = -\omega_{ji}$ $\quad \Pi_{ij} = -\Pi_{ji}$

(2) $$\frac{\partial \omega_{ij}}{\partial x_k} + \frac{\partial \omega_{jk}}{\partial x_i} + \frac{\partial \omega_{ki}}{\partial x_j} = 0$$

Let $A = (\Pi_{ij})$, $B = (\omega_{ij})$.

$$\Rightarrow \quad AB = I_n$$

$$\Rightarrow \quad \frac{\partial}{\partial x_\alpha}(AB) = \frac{\partial}{\partial x_\alpha}(I_n) = 0$$

$$\parallel$$

$$\left(\frac{\partial}{\partial x_\alpha} A\right) \cdot B + A \cdot \left(\frac{\partial B}{\partial x_\alpha}\right)$$

$$\Rightarrow \quad \frac{\partial A}{\partial x_\alpha} = - A \cdot \frac{\partial B}{\partial x_\alpha} \cdot B^{-1} = - A \frac{\partial B}{\partial x_\alpha} \cdot A$$

$$\Rightarrow \quad \frac{\partial \Pi_{ab}}{\partial x_\alpha} = - \sum_{\beta, \gamma} \Pi_{a\beta} \frac{\partial \omega_{\beta\gamma}}{\partial x_\alpha} \Pi_{\gamma b}$$

$$= + \sum_{\beta, \gamma} \Pi_{a\beta} \Pi_{b\gamma} \frac{\partial \omega_{\beta\gamma}}{\partial x_\alpha}$$

$$\Rightarrow \quad \sum_\alpha \Pi_{a\alpha} \frac{\partial \Pi_{bc}}{\partial x_\alpha} = \sum_{\alpha, \beta, \gamma} \Pi_{a\alpha} \Pi_{b\beta} \Pi_{c\gamma} \frac{\partial \omega_{\beta\gamma}}{\partial x_\alpha}$$

$$\Rightarrow \quad \underbrace{\sum_\alpha \left(\Pi_{a\alpha} \frac{\partial \Pi_{bc}}{\partial x_\alpha} + \Pi_{b\alpha} \frac{\partial \Pi_{ca}}{\partial x_\alpha} + \Pi_{c\alpha} \frac{\partial \Pi_{ab}}{\partial x_\alpha} \right)}_{= 0 \text{ by Jacobi}}$$

$$= \sum_{\alpha, \beta, \gamma} \Pi_{a\alpha} \Pi_{b\beta} \Pi_{c\gamma} \left(\frac{\partial \omega_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial \omega_{\gamma\alpha}}{\partial x_\beta} + \frac{\partial \omega_{\alpha\beta}}{\partial x_\gamma} \right)$$

π nondegenerate

0

$$\boxed{} = 0 \Rightarrow d\omega = 0$$

#

<u>Cor</u>

A mfd is symplectic

i.e. it's equipped with a non-degenerate closed 2-form

$\Leftrightarrow$ it's equipped with a non-degenerate Poisson str.