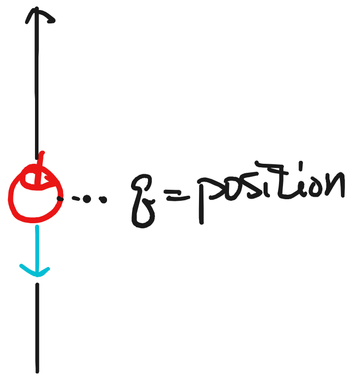
Geometric models for noncommutative algebras
Fall 2025, week 5

Poisson structures

Motivation: classical mechanics

Example



$$(*)\quad -mg = F = ma = m\ddot{q}$$

Introduce a new variable:

$$p := m\dot{q} \quad \text{(momentum)}$$

$$(*) \Rightarrow \quad (**) \begin{cases} \dot{q} = \dfrac{p}{m} = \dfrac{\partial H}{\partial p} \\[2ex] \dot{p} = m\ddot{q} = -mg = -\dfrac{\partial H}{\partial q} \end{cases}$$

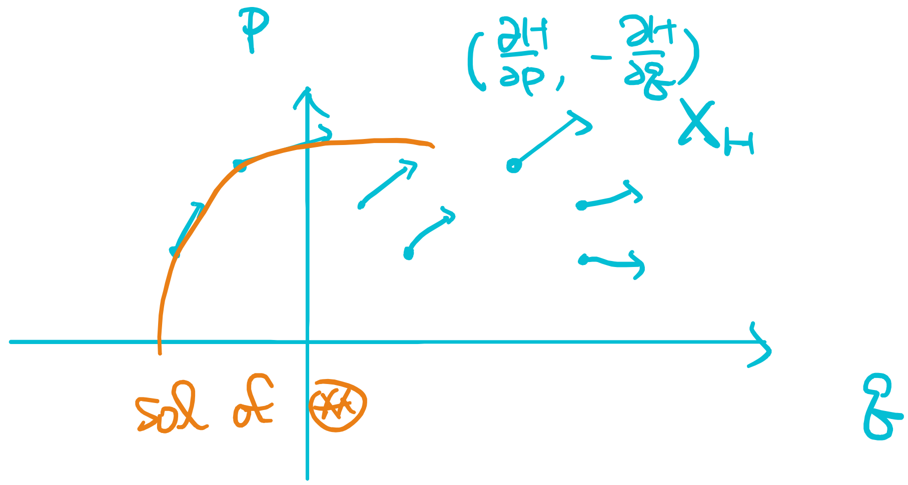
where $\dfrac{d}{dt}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}$

$$H(q,p) := \frac{p^2}{2m} + mgq$$

$$\underbrace{\qquad}_{\substack{\| \\ \frac{m^2v^2}{2m} = \frac{mv^2}{2}}} \qquad p = mv$$

(**) is the integral curve eq. of

$$X_H := \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$



Note that $\exists$ a "Poisson structure" behind X_H:

$$\{-,-\} : C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$$

$$\{f, g\} := \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}$$

Then

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

$$= \{H, -\}$$

So people consider $\{-,-\}$ as a structure that governs classical mechanics.

It is easy to check that $\{-,-\}$ satisfies:

Lie bracket

(i) skew-symmetry:

$$\{f, g\} = -\{g, f\}$$

(ii) Jacobi identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$$

(iii) Leibniz rule:

$$\{f\cdot g, h\} = f\{g,h\} + g\{f,h\}$$

$\forall\ f, g, h \in C^\infty(\mathbb{R}^2)$

Def

A commutative alg A is a Poisson algebra if $\exists$ Lie bracket

$$\{-,-\} : A \times A \to A$$

satisfying the Leibniz rule.

Such a bracket $\{ , \}$ is called a Poisson bracket or Poisson structure on A.

A manifold M is a Poisson manifold if $C^\infty(M)$ is equipped with a Poisson algebra structure.

e.g. $(\mathbb{R}^2, \{ , \})$ is a Poisson mfd.

Poisson structures on $\mathbb{R}^n$

Prop

If $\{-,-\}$ is a Poisson structure $\mathbb{R}^n$, then

① $$\{f,g\} = \sum_{i,j=1}^{n} \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \quad \forall f,g \in C^\infty(\mathbb{R}^n),$$

for some $(\Pi_{ij}) \in M_{n\times n}(C^\infty(\mathbb{R}^n))$ satisfying

② $$\Pi_{ij} = -\Pi_{ji}$$

and

③ $$\sum_{l=1}^{n} \left(\frac{\partial \Pi_{ij}}{\partial x_l} \Pi_{lk} + \frac{\partial \Pi_{jk}}{\partial x_l} \Pi_{li} + \frac{\partial \Pi_{ki}}{\partial x_l} \Pi_{lj} \right) = 0$$

for $i,j,k = 1,\cdots,n$.

Conversely, any $(\Pi_{ij}) \in M_{n\times n}(C^\infty(\mathbb{R}^n))$ satisfying ② and ③ defines a Poisson str. on $\mathbb{R}^n$ by ①.

pf

Step 1

Let $p \in \mathbb{R}^n$. By the Taylor expansion of f at p,

$\exists\, r_{ij} \in C^\infty(\mathbb{R}^n)$ s.t.

$$f(x) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\bigg|_p \cdot (x_i - p_i)$$

$$+ \sum^{n} (x_i - p_i)(x_j - p_j) \cdot r_{ij}(x)$$

$$\Rightarrow \{f,g\} = f(p)\cdot\{1,g\} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_p \cdot \{x_i - p_i, g\}$$

$$+ \sum_{i,j=1}^{n} \underline{\{(x_i-p_i)(x_j-p_j)\, r_{ij}(x),\ g\}}$$

$$\{1\cdot 1, g\}$$
$$= \{1,g\}\cdot 1 + 1\cdot\{1,g\}$$
$$= 2\{1,g\}$$
$$\Rightarrow \{1,g\} = 0$$

$$(\ \ )\Big|_{x=p}$$

$$= \{(x_i-p_i), g\}\, \underbrace{(x_j-p_j)}_{\to 0}\, r_{ij}(x)\Big|_{x=p}$$

$$+ \underbrace{(x_i-p_i)}_{\to 0}\cdot \{(x_j-p_j)\, r_{ij}(x), g\}\Big|_{x=p}$$

$$= 0$$

$$\Rightarrow \{f,g\}\Big|_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_p \cdot \{x_i - p_i,\ g\}\Big|_p$$

Apply the same argument to g with Taylor expansion of g:

$$\{f,g\}\Big|_p = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\Big|_p \cdot \frac{\partial g}{\partial x_j}\Big|_p \cdot \{x_i, x_j\}\Big|_p$$

Let

$$\Pi_{ij} = \{x_i, x_j\} \in C^\infty(\mathbb{R}^n)$$

$$\Rightarrow \quad \{f,g\} = \sum_{i,j=1}^{n} \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \qquad \text{(A)}$$

Conversely, if Π_{ij} is an arbitrary $C^\infty(\mathbb{R}^n)$-valued $n \times n$ matrix, then (A) defines a bilinear map

$$\{-,-\} : C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

satisfying the Leibniz rule.

In other words, we have a 1-1 corr.

$$\left\{ \begin{array}{l} \{-,-\} : C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n) \\ \text{bilinear, and} \\ \quad \{fg,h\} = \{f,h\}g + f\{g,h\} \\ \quad \{f,gh\} = \{f,g\}h + g\{f,h\} \end{array} \right\}$$

$\{-,-\}$

$$\text{1-1} \Big\uparrow \text{onto}$$

$$\{f,g\} = \sum_{i,j} \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

$$\left(\Pi_{ij} = \{x_i, x_j\}\right) \in M_{n\times n}(C^\infty(\mathbb{R}^n)) \ni (\Pi_{ij})$$

<u>Step 2</u>: Skew-Symmetry.

$$\{f,g\} + \{g,f\} = 0 \quad \forall f,g \in C^\infty(\mathbb{R}^n)$$

(A)

$$\Leftrightarrow \sum_{i,j=1}^n \Pi_{ij} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} + \sum_{i,j=1}^n \Pi_{ij} \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j} = 0 \quad \forall f,g \in C^\infty(\mathbb{R}^n)$$

$$= \sum_{i,j=1}^n \Pi_{ji} \frac{\partial g}{\partial x_j}\frac{\partial f}{\partial x_i}$$

$$\| $$

$$\sum_{i,j=1}^n (\Pi_{ij} + \Pi_{ji}) \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

$$\Leftrightarrow \quad \Pi_{ij} + \Pi_{ji} = 0 \quad \forall i,j = 1,\dots,n.$$

<u>Step 3</u> Jacobi id.

Consider the Jacobiator

$$J(f,g,h) := \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}$$

<u>exer</u>:

$$J(f_1 f_2, g, h) = f_1 \cdot J(f_2,g,h) + f_2 \cdot J(f_1,g,h)$$

and similar eq. also hold for 2nd and 3rd arguments

Then by the Taylor expansion argument,

$$J(f,g,h) = \sum_{i,j,k=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial x_k} \cdot J(x_i, x_j, x_k)$$

So

$$J(f,g,h) = 0 \qquad \forall f,g,h \in C^\infty(\mathbb{R}^n)$$

$$\Longleftrightarrow \quad J(x_i, x_j, x_k) = 0 \qquad \forall i,j,k = 1, \cdots, n$$

$$\|$$

$$\{\underbrace{\{x_i, x_j\}}_{\pi_{ij}}, x_k\} + \{\underbrace{\{x_j, x_k\}}_{\pi_{jk}}, x_i\} + \{\underbrace{\{x_k, x_i\}}_{\pi_{ki}}, x_j\}$$

$$\| \ (A)$$

$$\vdots$$

$$\|$$

$$\sum_{l=1}^{n} \left(\pi_{lk} \frac{\partial \pi_{ij}}{\partial x_l} + \pi_{li} \frac{\partial \pi_{jk}}{\partial x_l} + \pi_{lj} \frac{\partial \pi_{ki}}{\partial x_l} \right)$$

$\#$

Example (Constant Poisson str)

If (π_{ij}) is a constant matrix, then

③ is automatically satisfied.

So any skew-symmetric $n \times n$ matrix induces a constant Poisson str on $\mathbb{R}^n$.

For example, $(x_1, x_2) = (q, p)$

$$(\Pi_{ij})_{2\times 2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rightsquigarrow \quad \{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}$$

<u>Example</u> (Linear Poisson str)

If $\Pi_{ij} = \sum_{k=1}^{n} c_{ij}^{k} x_k$ for some constants c_{ij}^{k},

then

$\Pi_{ij}^{k} = -\Pi_{ji}^{k}$

$$② \Leftrightarrow \quad c_{ij}^{k} = -c_{ji}^{k} \qquad \forall\, i, j, k = 1, \cdots, n$$

$$③ \quad \sum^{n} \Big(\Pi_{l\,\cdot} \frac{\partial \Pi_{ij}}{\cdot} + \Pi \cdots + \Pi \cdots \Big) = 0$$

$$\Leftrightarrow \sum_{l=1}^{n} \left(C_{ij}^{l} C_{lk}^{m} + C_{jk}^{l} C_{li}^{m} + C_{ki}^{l} C_{lj}^{m} \right) = 0$$

$$\forall\ i,j,k,m = 1,\cdots,n$$

In this case, the constants C_{ij}^{k} are precisely the structure constants of a Lie algebra.

<u>Prop</u>

For any positive integer n, $\exists$ bijection

$$\{n\text{-dimensional Lie algebras over } \mathbb{R}\}$$

$$\updownarrow \text{ 1-1 onto}$$

$$\{\text{linear Poisson structures on } \mathbb{R}^n\}$$

For example, $so(3) = \{ A \in M_3(\mathbb{R}) \mid A^T + A = 0,\ (tr(A) = 0) \}$

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \end{pmatrix}$$

$\updownarrow$

-b-c0)

$\{\cdot,\cdot\}$ on $\mathbb{R}^3$,

$$\{x,y\}=z, \quad \{y,z\}=x, \quad \{z,x\}=y$$

and

$$\Pi = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

Hamiltonia vector fields

Let $(M, \{\cdot,\cdot\})$ be a Poisson mfd.

Given any $H \in C^\infty(M)$, define

$$X_H : C^\infty(M) \to C^\infty(M)$$

$$X_H(f) := \{H, f\}$$

By the Liebniz's rule, this defines a vector field $X_H \in \mathfrak{X}(M)$, called the Hamiltonian vector field of H.

Remark

If $M = \mathbb{R}^n$,

$$\{f, g\} = \sum_{i,j=1}^{n} \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

then

$$X_H = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \Pi_{ij} \frac{\partial H}{\partial x_i} \right) \cdot \frac{\partial}{\partial x_j}$$

And its integral curve of is

$$\frac{d x_j(t)}{dt} = \sum_{i=1}^{n} \Pi_{ij} \frac{\partial H}{\partial x_i} \quad \leftrightarrow \quad \text{Hamiltonian system}$$

Prop

$$[X_f, X_g] = X_{\{f,g\}} \qquad \forall f, g \in C^\infty(M)$$

pf

$\forall h \in C^\infty(M)$,

$$[X_f, X_g](h) = X_f(X_g(h)) - X_g(X_f(h))$$

$$= \{f, \{g, h\}\} - \{g, \{f, h\}\}$$

$$= + \{g, \{h, f\}\}$$

$\{g,\{h,f\}\}$

$$\overset{\text{Jacobi}}{=} -\{h,\{f,g\}\}$$

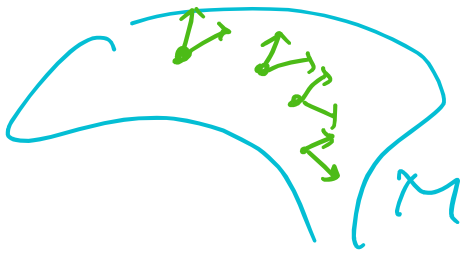
$$= \{\{f,g\},h\} = X_{\{f,g\}}(h). \qquad \#1$$

Schouten brackets of multivector fields

Let M be a mfd.

A *k-vector field* on M is a global section of

$$\Lambda^k TM \to M$$



So

$$\{0\text{-vector fields}\} = \Gamma(\Lambda^0 TM) = C^\infty(M)$$

$$\{1\text{-vector fields}\} = \Gamma(\Lambda^1 TM) = \mathfrak{X}(M)$$

$$\{2\text{-vector fields}\} = \Gamma(\Lambda^2 TM)$$

$\parallel$

bivector fields

Recall:

$$\Gamma(\Lambda TM) = \Lambda_{C^\infty(M)} \Gamma(TM)$$

$\leftarrow$ $X \wedge Y = -Y \wedge X$

$$= \coprod_{C^\infty(M)} (\Gamma(TM)) \Big/ \langle X \otimes Y + Y \otimes X \rangle_{X, Y \in \Gamma(TM)}$$

as $C^\infty(M)$-modules

So

$$\coprod_{C^\infty(M)} (\Gamma(TM)) \twoheadrightarrow \Gamma(\Lambda TM)$$

$$\cup \qquad\qquad\qquad \cup$$

$$\overbrace{\Gamma(TM) \otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} \Gamma(TM)}^{k} \twoheadrightarrow \Gamma(\Lambda^k TM)$$

<u>Prop</u>

$\exists !$ bilinear operation, called Schouten bracket,

$$[-,-] : \Gamma(\Lambda^a TM) \times \Gamma(\Lambda^b TM) \to \Gamma(\Lambda^{a+b-1} TM)$$

satisfying the following properties:

$\forall A \in \Gamma(\Lambda^a TM)$, $B \in \Gamma(\Lambda^b TM)$, $C \in \Gamma(\Lambda^c TM)$,

$$[A, B] = -(-1)^{(a-1)(b-1)} [B, A]$$

(4) $[A,B] = -(-1) \quad [B,A]$

(5) $[A, B\wedge C] = [A,B]\wedge C + (-1)^{(a-1)b} B\wedge[A,C]$

(6) $(-1)^{(a-1)(c-1)}[A,[B,C]] + (-1)^{(b-1)(a-1)}[B,[C,A]]$
$+ (-1)^{(c-1)(b-1)}[C,[A,B]] = 0$

which extends the following relations:

$[f,g] = 0 \qquad \forall f, g \in C^\infty(M),$

$[X,f] = X(f) \qquad \forall X \in \mathfrak{X}(M) = \Gamma(\Lambda^1 TM)$
$f \in C^\infty(M),$

$[X,Y] = [X,Y]_{Lie}$ = usual Lie bracket of vector fields
$\forall X, Y \in \mathfrak{X}(M).$

Remark

A ($\mathbb{Z}$-)graded Lie algebra is a vector sp. with a decomposition ← direct sum of vector spaces

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$$

bilinear

and a bracket

$$[\ ,\ ]: \mathfrak{g}_a \times \mathfrak{g}_b \to \mathfrak{g}_{a+b}$$

s.t.

(i) $[A,B] = -(-1)^{ab}[B,A]$

(ii) $[A,[B,C]] = [[A,B],C] + (-1)^{ab}[B,[A,C]]$

$\forall\ A \in \mathfrak{g}_a,\ B \in \mathfrak{g}_b,\ C \in \mathfrak{g}_c$

Let

$$\mathfrak{g}_i = \Gamma(\Lambda^{i+1} TM)$$

Then by ④ and ⑥,

$$\mathfrak{g} = \bigoplus_{i=-1}^{\infty} \mathfrak{g}_i = \bigoplus_{i=-1}^{\infty} \Gamma(\Lambda^{i+1} TM)$$

is a ($\mathbb{Z}$-) graded Lie algebra.

Example

On $\mathbb{R}^3$. ← coordinate: (x,y,z)

$\in \Gamma(\Lambda^2 TM)$

① $\left[\underbrace{\frac{\partial}{\partial x}}_{\in\, \Gamma(\Lambda^1 TM)}, \ \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \right]$

$$\overset{⑤}{=} \underbrace{\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]_{Lie}}_{=\,0} \wedge \frac{\partial}{\partial z} + (-1)^{(1-1)\cdot 1} \frac{\partial}{\partial y} \wedge \underbrace{\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right]_{Lie}}_{=\,0}$$

$$= \underbrace{0}_{\in\, \Gamma(\Lambda^2 TM)}$$

② $\left[\underbrace{\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}}_{\in\, \Gamma(\Lambda^2 TM)}, \ \underbrace{x \cdot \frac{\partial}{\partial z}}_{\in\, \Gamma(\Lambda^1 TM)} \right]$

$$\overset{④}{=} -(-1)^{(2-1)(1-1)} \left[x \cdot \frac{\partial}{\partial z}, \ \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \right]$$

$$\overset{⑤}{=} -\left(\underbrace{\left[x \frac{\partial}{\partial z}, \frac{\partial}{\partial x} \right]}_{=\,-\frac{\partial}{\partial z}} \wedge \frac{\partial}{\partial y} + (-1)^{(1-1)\cdot 1} \frac{\partial}{\partial x} \wedge \underbrace{\left[x \frac{\partial}{\partial z}, \frac{\partial}{\partial y} \right]}_{=\,0} \right)$$

$$= \underbrace{\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial y}}_{\Gamma(\Lambda^2 TM)} \quad \# \quad \Gamma(\Lambda^2 TM)$$

③ $\left[\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y},\ \frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right]$

$$\overset{⑤}{=} \underbrace{\left[\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y},\ \frac{\partial}{\partial y}\right]}_{\overset{\text{similar ②}}{=}\ 0}\wedge\frac{\partial}{\partial z} + (-1)^{(2-1)\cdot 1}\frac{\partial}{\partial y}\wedge\underbrace{\left[\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right]}_{\overset{\text{similar to ②}}{=}\ 0}$$

$$= 0$$

<u>Lemma</u>

If

$$\Pi = \frac{1}{2}\sum_{i,j=1}^{n}\Pi_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} \in \Gamma(\Lambda^2 T\mathbb{R}^n)$$

and $f \in C^\infty(\mathbb{R}^n)$

then

$$[\Pi, f] = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\Pi_{ji}\right)\frac{\partial}{\partial x_i}$$

$$\left(\overset{\text{if } \Pi = \text{Poisson}}{=} \quad X_f\right)$$

$$[\Pi, \Pi]$$

$$\Big(\quad n \qquad\qquad\qquad\qquad \Big)$$

$$= \sum_{1\le i<j<k\le n} \left(\sum_{l=1}^{n} \left(\Pi_{kl} \frac{\partial \Pi_{ij}}{\partial x_l} + \Pi_{il} \frac{\partial \Pi_{jk}}{\partial x_l} + \Pi_{jl} \frac{\partial \Pi_{ki}}{\partial x_l} \right) \right.$$

$$\left. \cdot \quad \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} \right.$$