Geometric models for noncommutative algebras

Fall 2025, week 4

Left invariant differential operators

Recall

① Let G be a matrix gp. We have

$$\mathfrak{g} = T_e G = \{\text{deriations at } e \in G\}$$

$$\updownarrow$$

$$\mathfrak{X}^L(G) = \{\text{left invariant vec. fields on } G\}$$

② A vector field X is locally of the form

diff. op. of order one

$$X \overset{\text{locally}}{=} \sum_{i=1}^{n = \dim G} a_i(x) \cdot \frac{\partial}{\partial x_i}$$

What if we consider operators

locally of the form

locally of the form

$$D \overset{\text{locally}}{=} \sum_{I} a_I(x) \frac{\partial^{|I|}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}$$

where

$$I = (i_1, i_2, \cdots, i_n) \in (\mathbb{Z}_{\geq 0})^n$$

$$|I| = i_1 + i_2 + \cdots + i_n$$

e.g.

$$D = a(x) \frac{\partial^2}{\partial x_1 \partial x_2} + b(x) + c(x) \frac{\partial}{\partial x_1}$$

$$\underset{C^\infty(\mathbb{R}^2)}{\underset{\rotatebox{90}{$\in$}}{f}} \xmapsto{D} a(x) \frac{\partial^2 f}{\partial x_1 \partial x_2} + b(x) \cdot f(x) + c(x) \cdot \frac{\partial f}{\partial x_1}$$

We expect:

$$D_e(G) = \left\{ \begin{array}{c} \text{differential operators} \\ \text{at } e \in G \end{array} \right\}$$

$$\updownarrow$$

$$D^L(G) = \left\{ \begin{array}{c} \text{left invariant diff.} \\ \text{operators on } G \end{array} \right\}$$

Def

For $f \in C^\infty(G)$, define

order-zero differential operator

$$m_f \; (=f) : C^\infty(G) \to C^\infty(G),$$
$$h \longmapsto m_f(h) = f \cdot h$$

alg w.r.t. composition $\circ$

$$D(G) := \text{subalg of } \underbrace{\mathrm{End}_{\mathbb{k}}(C^\infty(G))}$$
generated by
$$m_f, \quad f \in C^\infty(G)$$
$$X, \quad X \in \mathfrak{X}(G)$$

That is, an element in $D(G)$ is of the form

$$\sum X_1 \circ m_{f_1} \circ X_2 \circ X_3 \circ m_{f_2} \circ X_4 : C^\infty(G) \to C^\infty(G)$$

Such an operactor is called a <u>(linear) differential operator</u> on G

A differential operator on G at $g \in G$.

is a linear map $u : C^\infty(G) \to \mathbb{k}$ s.t.

$\exists\, D \in \mathcal{D}(G)$ with the property

$$u(f) = \underbrace{(D(f))}_{\in\, C^\infty(G)}(g) \qquad \forall f \in C^\infty(G)$$

Define

$$\mathcal{D}^L(G) = \begin{array}{c}\text{subalg of } \operatorname{End}_{\mathbb{k}}(C^\infty(G)) \\ \text{generated by } \mathfrak{X}^L(G)\end{array}$$

A element $D \in \mathcal{D}^L(G)$ is called

a left invariant differential operator

on G.

exer

A diff. op. $D \in \mathcal{D}(G)$ is left invariant

$\iff \quad D(f \circ L_g) = D(f) \circ L_g \quad \forall g \in G \ \forall f \in C^\infty(G)$

similar to $\mathfrak{X}^L(G) \cong T_eG = \mathfrak{g}$

Prop

The map

$$D^L(G) \longrightarrow D_e(G), \quad D \mapsto D_e$$

is an iso of vector sp, where

$$D_e : C^\infty(G) \to \mathbb{k}, \quad D_e(f) = (D(f))(e)$$

Remark

A different operator $D \in D(G)$ is local

That is, $\forall g \in G$, if $f_1, f_2 \in C^\infty(G)$ and $\exists$ neighborhood U of g s.t. $f_1|_U = f_2|_U$, then

$$(D(f_1))(g) = (D(f_2))(g)$$

$\Rightarrow$ $D(f)(g)$ depends only on $f|_U$, where U is any neighborhood of $g \in G$

$\Rightarrow$ D admits a representation in local

Coordinates.

In a local chart $U \ni x = (x_1, \dots, x_n)$

$$D = \sum_{I \in (\mathbb{Z}_{\geq 0})^n} \underbrace{a_I(x)}_{\in C^\infty(U)} \frac{\partial^{|I|}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}$$

only finitely many of $a_I(x)$ are nonzero

The order of D is the smallest number $m \geq 0$ s.t. in any local chart,

$$a_I(x) = 0 \quad \forall\ |I| > m$$

Similarly, for $D_e \in D_e(G)$

$$D_e \overset{\text{locally}}{=} \sum_{I \in (\mathbb{Z}_{\geq 0})^n} c_I \frac{\partial^{|I|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$$

where $c_I \in \mathbb{k}$, and only finitely many are nonzero.

Recall

tensor alg of $\mathfrak{g}$. $\bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$

$$\mathcal{U}(\mathfrak{g}) = T\mathfrak{g} \Big/ \langle X\otimes Y - Y\otimes X - [X,Y] : X, Y \in \mathfrak{g} \rangle$$

Identify $\mathfrak{g}$ with $\mathfrak{X}^L(G)$, we have

$X_i \in \mathfrak{g} = \mathfrak{X}^L(G) \subset D^L(G)$

$$\mathcal{U}(\mathfrak{g}) \to D^L(G), \quad X_1 X_2 \cdots X_k \mapsto X_1 \circ X_2 \circ \cdots \circ X_k$$

well-defined, since $X\otimes Y - Y\otimes X - [X,Y]$

$$\mapsto X\circ Y - Y\circ X - \underbrace{[X,Y]}_{\substack{\parallel \\ X\circ Y - Y\circ X}} = 0$$

Furthermore,

Prop

The map $\mathcal{U}(\mathfrak{g}) \to D^L(G)$ is an iso of alg.

$D^L(G) \cong D_e(G) = \{ C^\infty(G) \to \mathbb{k} \text{ s.t. } \cdots \}$

This identification leads to the consideration of comultiplication on $\mathcal{U}(\mathfrak{g})$: $C^\infty(G) \otimes C^\infty(G) \to C^\infty(G\times G)$

$D_1 \otimes D_2$

$D^L(G) \to D^L(G) \otimes D^L(G)$ $(D_1 \otimes D_2)(f_1 \otimes f_2)$

$$\Delta : D(G) \to \underbrace{D(G) \otimes D(G)}$$

$$D_1(f_1) \cdot D_2(f_2)$$

$$\underline{\Delta(D)} \, (\underbrace{f_1 \otimes f_2}_{C^\infty(G) \otimes C^\infty(G)}) := D(\underbrace{f_1 \cdot f_2}_{C^\infty(G)})$$

$$C^\infty(G) \otimes C^\infty(G) \to C^\infty(G)$$

e.g. For $X \in \mathfrak{X}^L(G)$,

$$(X \otimes id)(f_1 \otimes f_2)$$

$$(\Delta X)(\underline{f_1 \otimes f_2}) = X(f_1 \cdot f_2) = \underline{X(f_1) f_2} + \underline{f_1 X(f_2)}$$

$$(id \otimes X)(f_1 \otimes f_2)$$

$$= (X \otimes id + id \otimes X)(\underline{f_1 \otimes f_2})$$

$$\Rightarrow \quad \Delta X = X \otimes id + id \otimes X$$

This defines a comultiplication on $\mathcal{U}(\mathfrak{g})$:

$$\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}).$$

which is determined by

$$① \begin{cases} \Delta(1) = 1 \otimes 1 \\ \Delta(X) = X \otimes 1 + 1 \otimes X \quad \forall X \in \mathfrak{g} \\ \Delta(u \cdot v) = \underline{\Delta(u)} \cdot \underline{\Delta(v)} \end{cases}$$

$\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$

$$(u_1 \otimes u_2) \cdot (v_1 \otimes v_2) = (u_1 v_1) \otimes (u_2 \cdot v_2)$$

e.g.

$$\Delta(X \cdot Y) = \Delta(X) \cdot \Delta(Y)$$

$$= (1 \otimes X + X \otimes 1) \cdot (1 \otimes Y + Y \otimes 1)$$

$$= 1 \otimes X \cdot Y + Y \otimes X + X \otimes Y + X \cdot Y \otimes 1$$

In general,

$$\Delta(\underbrace{X_1 \cdot X_2 \cdots X_k}_{\in\, \mathcal{U}\mathfrak{g}}) = 1 \otimes (X_1 \cdots X_k) + (X_1 \cdots X_k) \otimes 1 + \sum_{\substack{\sigma \in S_{p,q} \\ p+q=k \\ p,q>0}} (X_{\sigma(1)} \cdots X_{\sigma(p)}) \otimes (X_{\sigma(p+1)} \cdots X_{\sigma(p+q)}) \quad ②$$

where $S_{p,q} = \left\{ \sigma : \underset{\text{1-1, onto}}{\{1, \cdots, p+q\} \circlearrowleft} \;\middle|\; \begin{array}{l} \sigma(1) < \sigma(2) < \cdots \sigma(p), \\ \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q) \end{array} \right\}$

~~ex~~

① $\Leftrightarrow$ ②.

PBW map and exponential map

Let $\mathfrak{g}$ be a finite-dimensional Lie alg. Consider the Poincaré-Birkhoff-Witt map

$$T\mathfrak{g} / \langle X\otimes Y - Y\otimes X : X, Y \in \mathfrak{g} \rangle$$

$$\text{pbw}: \underset{\parallel}{S(\mathfrak{g})} \longrightarrow \mathcal{U}(\mathfrak{g})$$

$$\text{pbw}(X_1 \odot \cdots \odot X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} X_{\sigma(1)} \cdot X_{\sigma(2)} \cdots X_{\sigma(k)}$$

where S_k denotes the symmetric gp of deg k.

$$S_k = \left\{ \{1, \ldots, k\} \xrightarrow[\text{onto}]{1-1} \{1, \ldots, k\} \right\}$$

It is a classical algebraic result that pbw is a vector space iso. (PBW Thm)

Now: a geometric point of view of pbw: $S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$.

Let G be a matrix group, $\mathfrak{g} = \mathrm{Lie}(G)$.

$\mathfrak{g}_0 = \mathrm{Lie}(\mathfrak{g}, +) =$ the Lie alg with
vec. sp. $= \mathfrak{g}$
Lie bracket $= [\,,\,]_0$
$[X, Y]_0 = 0 \quad \forall X, Y \in \mathfrak{g}$

$$\Rightarrow \mathcal{U}(\mathfrak{g}_0) = T\mathfrak{g} \Big/ \langle \underbrace{X \otimes Y - Y \otimes X} - \underbrace{\cancel{[X,Y]_0}}_{=0} : X, Y \in \mathfrak{g} \rangle$$

$$= S(\mathfrak{g})$$

$$\Rightarrow S(\mathfrak{g}) = \underset{\substack{= \\ e=0 \text{ in } \mathfrak{g}}}{D_0}(\mathfrak{g}) = \{ \underline{C^\infty(\mathfrak{g})} \to \Bbbk \mid \ldots \}$$

$\exp_* \downarrow \qquad \uparrow \exp^*$

$$\mathcal{U}(\mathfrak{g}) = \underset{e \in G}{D_e}(\mathfrak{g}) = \{ \underline{C^\infty(G)} \to \Bbbk \mid \ldots \}$$

Consider

$$\exp : \mathfrak{g} \longrightarrow G, \quad \exp(A) = \sum_{k=1} \frac{A^k}{k!}$$

$$\rightsquigarrow \quad \exp^* : C^\infty(G) \to C^\infty(\mathfrak{g}), \quad f \mapsto \underbrace{f \circ \exp}_{\substack{\text{on alg. mor.}}} = \exp^*(f)$$

$$\begin{tikzcd} & & \mathbb{k} \\ \mathfrak{g} \arrow[rr, "\exp"'] \arrow[urr, dashed, "f\circ\exp = \exp^*(f)"] & & G \arrow[u, "f"'] \end{tikzcd}$$

$$\rightsquigarrow \quad \exp_* : \underset{D}{\overset{S(\mathfrak{g})}{D_0(\mathfrak{g})}} \longrightarrow \overset{U(\mathfrak{g})}{D_e(G)},$$

$$(\exp_* D)(\underset{C^\infty(G)}{f}) := D(\exp^*(f)) = D(\underset{C^\infty(\mathfrak{g})}{f \circ \exp})$$

__Claim__:

$$\mathrm{pbw} = \exp_* : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g}).$$

__Lemma__

The tangent map

$$\overline{T_0 \exp} = \mathrm{id}_{\mathfrak{g}} : \overline{T_0 \mathfrak{g}} \cong \mathfrak{g} \longrightarrow \overline{T_e G} = \mathfrak{g}$$

is the identity map

In particular, $\exists$ nbd $\underset{\text{open}}{U \subset \mathfrak{g}}$ of 0 and nbd $V \underset{\text{open}}{\subset} G$ of e st,

$$\exp|_U : U \longrightarrow V$$

is a diffeomorphism.

pf

Consider $A \in \mathfrak{g}$, $\gamma^A(t) = t \cdot A \in \mathfrak{g}$ st $\gamma^A(0) = 0$, $\gamma^{A\prime}(0) = A$

$$(T_0 \exp)(A) = (T_0 \exp)(\gamma^{A\prime}(0))$$

$$= (\exp \circ \gamma^A)'(0) = (\exp(tA))'\Big|_{t=0}$$

$$= A = \mathrm{id}_{\mathfrak{g}}(A) \qquad \#$$

Lemma

$$\exp_* : D_0(\mathfrak{g}) \to D_e(G)$$

is an iso of coalg.

Here a coalg mor is

$$\phi : (C, \Delta) \to (C', \Delta')$$

st, $\Delta' \circ \phi = (\phi \otimes \phi) \circ \Delta$

pf

Recall:

$C \xrightarrow{\Phi} C' \xrightarrow{\Phi'} C' \circ C'$ $\quad C \xrightarrow{\Phi} C \circ C \xrightarrow{\Phi \circ \Phi} C' \circ C'$

For $D_0 \in \mathcal{D}_0(\mathfrak{g})$, $D_0(f)$ depends only on $f|_U$, and for $D_e \in \mathcal{D}_e(G)$, $D_e(f')$ depends only on $f|_V$.

$\Rightarrow$ $\exp_* D_0$ is determined by

originally $C^\infty(G)$ $\updownarrow$

$$(\exp_* D_0)(f') = D_0(f' \circ \exp) \qquad f' \in \underline{C^\infty(V)}$$

$\cong$

$C^\infty(U)$

Since $\exp: U \to V$ is a diffeo, for each $f \in C^\infty(U)$, $\exists\, f' = f \circ \exp^{-1} \in C^\infty(V)$

$$f = \exp^*(f')$$

$\Rightarrow$ $\exp_*$ is an iso (of v.s.):

1-1: $(\exp_* D_0) = 0 \Rightarrow D_0(f) = (\exp_* D_0)(f \circ \exp^{-1}) = 0$

$f \mapsto D_e(f \circ \exp^{-1})$

onto: $\forall\, D_e \in \mathcal{D}_e(G)$, we have $\underline{(\exp^{-1})_* D_e}$ s.t.

$$\exp_*(\exp_*^{-1} D_e) = D_e.$$

note: "...... maps" alg mor

Furthermore, "$\exp_*$ is dual to $\exp^*$"

$$(\Delta\, \exp_*\, D_0)(f_1 \otimes f_2) = (\exp_* D_0)(f_1 \cdot f_2)$$

$$= D_0(\exp^*(f_1 \cdot f_2)) = D_0(\exp^*(f_1) \cdot \exp^*(f_2))$$

$$= (\Delta D_0)(\exp^* f_1 \otimes \exp^* f_2)$$

$$= ((\exp_* \otimes \exp_*) \circ (\Delta D_0))(f_1 \otimes f_2)$$

$\Rightarrow$ $\exp_*$ is a coalg mor. #

Lemma

$$\text{pbw} : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$$

is the unique coalg mor s.t.

$$\text{pbw}(a) = a \quad \forall\, a \in S^0(\mathfrak{g}) = \mathbb{k}$$

$$\text{pbw}(X) = X \quad \forall\, X \in S^1(\mathfrak{g}) = X$$

sketch of pf

Let $F : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ be a coalg mor s.t.

$F(a) = a$, $F(X) = X$. $\qquad S^0\mathfrak{g} \oplus S^1\mathfrak{g} \oplus \cdots \oplus S^k\mathfrak{g}$

Then we prove $F|_{S^k} = \text{pbw}|_{S^k}$

[illegible] $|_{S^{\le n}\mathfrak{g}} = \mathrm{pbw}|_{S^{\le n}\mathfrak{g}}$

by induction:

$n=1$ is ok by assumption

Assume $F|_{S^{<k}\mathfrak{g}} = \mathrm{pbw}|_{S^{<k}\mathfrak{g}}$. Then

$$\Delta(F(X_1 \odot \cdots \odot X_k)) = (F \otimes F)(\Delta(X_1 \odot \cdots \odot X_k))$$

$$= 1 \otimes F(X_1 \odot \cdots \odot X_k) + F(X_1 \odot \cdots \odot X_k) \otimes 1$$

$$+ \sum_{\substack{\sigma \in S_{p,q} \\ p+q=k \\ p,q>0}} \underset{\substack{\shortparallel \\ \mathrm{pbw}}}{F}(X_{\sigma(1)} \odot \cdots \odot X_{\sigma(p)}) \otimes \underset{\substack{\shortparallel \\ \mathrm{pbw}}}{F}(X_{\sigma(p+1)} \odot \cdots \odot X_{\sigma(p+q)})$$

induction hypothesis

and then compute .... (use basis for $\mathfrak{g}$).

...

#

By the lemmas, we have

Thm (Poincaré–Birkhoff–Witt)

Let G be a matrix gp, and $\mathfrak{g} = \mathrm{Lie}(G)$.

Then

[illegible]

$$\mathrm{pbw} = \exp_* : S(\mathfrak{g}) = U_0(\mathfrak{g}) \to D_e(G) \overset{\cong}{=} U(\mathfrak{g})$$

is an iso of coalg's.

Infinity jets

An infinity jet at $e \in G$ is essentially the Taylor expansion of a function on G at e.

Recall that if $f \in C^\infty(\mathbb{R}^n)$ and $p \in \mathbb{R}^n$, then the Taylor expansion of f at p is

$$\sum_{I \in (\mathbb{Z}_{\geq 0})^n} \frac{D_p^I(f)}{I!} (x-p)^I$$

where

$$x = (x_1, \dots, x_n)$$

$$I = (i_1, \dots, i_n) \in (\mathbb{Z}_{\geq 0})^n$$

$$I! = i_1! \cdot i_2! \cdots i_n!$$

$$D_p^I(f) = \frac{\partial^{|I|} f}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \Big|_{x=p}$$

$|I| = i_1 + i_2 + \cdots + i_n$

$p = (p_1, \dots, p_n) \in \mathbb{R}^n$

$$(x-p)^I = (x_1-p_1)^{i_1}(x_2-p_2)^{i_2} \cdots (x_n-p_n)^{i_n}$$

Note that the Taylor expansion of f is determined by the numbers $D_p^I(f)$, $I \in (\mathbb{Z}_{\geq 0})^n$.

Consider D_p^I as a variable, this set of numbers $\{ D_p^I(f) \mid I \in (\mathbb{Z}_{\geq 0})^n \}$ can be considered as the linear map:

$$D_p(\mathbb{R}^n) \longrightarrow \mathbb{R}, \quad D_p \mapsto D_p(f)$$

Motivated by this, we define

$$J(\mathfrak{g}) := \mathrm{Hom}_{\Bbbk}(\mathcal{U}(\mathfrak{g}), \Bbbk)$$

which is a model for Taylor series of functions on G at e.

<u>Remark</u>

$\exists$ a natural map

$$C^\infty(G) \xrightarrow{\tau} J(\mathfrak{g}), \quad f \mapsto \tau f,$$

where — Taylor series of f

where

$$\tau_f : \mathcal{U}(\mathfrak{g}) \longrightarrow \Bbbk, \quad \tau_f(u) = u(f)$$

e.g. $G = \mathbb{R}^n$

$$\underset{\mathcal{U}(\mathbb{R}^n)}{\overset{\cup}{D^I_{e=0}}} \xmapsto{\tau_f} \underset{\mathbb{R}}{\overset{\cup}{D^I_0(f)}} = \left.\frac{\partial^{|I|} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}\right|_{x=0}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ || a Taylor coeff of f

Remark

The comultiplication Δ on $\mathcal{U}(\mathfrak{g})$ induces a multiplication on $\mathcal{J}(\mathfrak{g}) = \mathrm{Hom}_{\Bbbk}(\mathcal{U}(\mathfrak{g}), \Bbbk)$:

For $\xi, \eta \in \mathcal{J}(\mathfrak{g})$, $u \in \mathcal{U}(\mathfrak{g})$, "convolution product"

$$\langle \xi \cdot \eta \mid u \rangle := \langle \xi \otimes \eta \mid \Delta u \rangle$$

Here

$$\langle - \mid - \rangle : \mathcal{J}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \to \Bbbk$$

$$\langle \xi \mid u \rangle = \xi(u)$$

and

$$\langle \ \mid \ \rangle : \mathcal{J}(\mathfrak{g}) \otimes \mathcal{J}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \to \Bbbk$$

$$\langle \xi \otimes \eta \mid u \otimes v \rangle = \xi(u) \cdot \eta(v).$$

An alternative approach:

Consider the Taylor series as the limit of

$$S_k = \sum_{\substack{I \in (\mathbb{Z}_{\geq 0})^n \\ |I| \leq k}} \frac{D_p^I(f)}{I!} (x-p)^I$$

To develop this approach on G without coordinates, consider

$$I_e := \{ f \in C^\infty(G) \mid f(e) = 0 \}$$

and

$$I_e^k = \{ f_1 \cdot f_2 \cdots f_k \mid f_1, \dots, f_k \in I_e \}$$

$\Rightarrow$ We have a descending seq. of ideals:

$$C^\infty(G) \supseteq I_e \supseteq I_e^2 \supseteq I_e^3 \supseteq \cdots \supseteq I_e^k \supseteq I_e^{k+1} \supseteq \cdots$$

$$\Rightarrow \quad \overset{S_1 \in}{C^\infty(G)/I_e} \twoheadleftarrow \overset{S_2 \in}{C^\infty(G)/I_e^2} \twoheadleftarrow \overset{S_3 \in}{C^\infty(G)/I_e^3} \twoheadleftarrow \cdots$$

$\ni (x_i - e_i)$

$$\sum_I \frac{D_e^I(f)}{I!} (x-e)^I = f(e) \text{ in}$$

$$\sum_I \frac{D_e^I(f)}{I!} (x-e)^I = \sum_{|I| \leq 1} \frac{D_e^I(f)}{I!} (x-e)^I$$

Consider the projective limit:

$$\varprojlim_k \frac{C^\infty(G)}{I_e^k} = \left\{ (S_k)_{k=1}^{\infty} \in \prod_{k=1}^{\infty} \frac{C^\infty(G)}{I_e^k} \;\middle|\; S_{k+1} \mapsto S_k \right\}$$

Remark

$$\tilde{\tau} : C^\infty(G) \longrightarrow \varprojlim_k \frac{C^\infty(G)}{I_e^k}, \quad f \mapsto \tilde{\tau}_f$$

where

$$\tilde{\tau}_f = (\tilde{\tau}_{f,k})_{k=1}^{\infty}, \quad \text{and}$$

$$\tilde{\tau}_{f,k} = \Big[\sum_{\substack{I \in (\mathbb{Z}_{\geq 0})^n \\ |I| < k}} \frac{D_e^I(f)}{I!} (x-e)^I \Big]$$

$$\uparrow$$

$$\frac{C^\infty(G)}{I_e^k}$$

Prop

The map

$$\varprojlim_k \frac{C^\infty(G)}{I_e^k} \xrightarrow{\Phi} J(\mathfrak{g})$$

$$S = (S_k) \longmapsto \Phi_S$$

is an iso of algebras, where

$$\Phi_s : \mathcal{U}(\mathfrak{g}) \to k$$

$$\langle \Phi_s \mid u \rangle = u(s_k)$$

if $u \in \underbrace{\mathcal{U}^{\leq k}(\mathfrak{g})}$. $=$ the image of $\bigoplus_{i=0}^{k} \mathfrak{g}^{\otimes i}$

in $T\mathfrak{g} \big/ \langle X\otimes Y - Y\otimes X - [X,Y] \rangle$ $= \mathcal{U}(\mathfrak{g})$

exer: prove the proposition

Thm

$$\text{pbw} = \underline{\exp_*} : S(\mathfrak{g}) \cong D_0(\mathfrak{g}) \longrightarrow D_e(G) \cong \mathcal{U}(\mathfrak{g})$$

is an iso of coalgebras

Pf Step 1:

We first compute $\exp_*(A^{\otimes k})$, $A \in \mathfrak{g}$. $\in S^k(\mathfrak{g})$

Note $A \in \mathfrak{g}$ generates a left invariant vec. field on $\mathfrak{g}$: $Y^A \in \mathfrak{X}^L(\mathfrak{g})$, $Y^A_B = \frac{d}{dt}\Big|_{t=0} (B + tA)$

$\Rightarrow$ for $f \in C(\mathfrak{g})$, $\quad A \rightarrow$

$$Y^A(f') \in C^\infty(\mathfrak{g}), \quad Y^A(f')\big|_B = Y^A_B(f') = \frac{d}{dt}\Big|_{t=0} f'(B+tA)$$

Y^A_0

$$\forall f \in C^\infty(G),$$

$$\exp_*(A^{\otimes k})(f) = A^{\otimes k}(f\circ\exp)$$

$$= \overbrace{Y^A \circ Y^A \cdots \circ Y^A}^{k \text{ times}}(f\circ\exp)\Big|_0$$

$$= (Y^A)^{k-1}\left(\frac{d}{dt_1}\Big|_{t_1=0}(f\circ\exp)(\square + t_1A)\right)\Big|_0$$

$$= (Y^A)^{k-2}\left(\frac{d}{dt_2}\Big|_{t_2=0}\left(\frac{d}{dt_1}\Big|_{t_1=0}(f\circ\exp)(\square + t_2A + t_1A)\right)\right)\Big|_0$$

$$= \cdots$$

$$= \frac{d}{dt_k}\Big|_{t_k=0} \cdots \frac{d}{dt_1}\Big|_{t_1=0}\left((f\circ\exp)(\overset{0}{\cancel{\square}} + t_kA + \cdots + t_1A)\right)\Big|_0$$

$$\mathrm{pbw}(A^{\otimes k})(f) = ?$$

$$\| \quad X_1 = \cdots X_k = A$$

$$\frac{1}{!}\sum A\cdot\cdots\cdot A = A^k \leftarrow$$

meaning: $A \leftrightarrow X^A$

$X^A_g = g\cdot A$

k-time

$k;\ \sigma \in S_k$

$(X^A \circ \cdots \circ X^A)(f)\big|_e$

Note: the flow of X^A is

$\Phi_t : G \to G,\quad \Phi_t(g) = g \cdot \exp(tA)$

$$\Rightarrow (X^A)^k (f)\big|_e = \left((X^A)^{k-1} \left(\frac{d}{dt_1}\Big|_{t_1=0} f(\underbrace{\Phi_t(-)}_{-\cdot \exp(t_1 A)}) \right) \right)\Big|_e$$

$= \cdots$

$$= \left(\frac{d}{dt_k}\Big|_{t_k=0} \cdots \frac{d}{dt_1}\Big|_{t_1=0} \right) \left(f(\square \cdot \exp(t_k A) \cdot \exp(t_{k-1} A) \cdots \exp(t_1 A)) \right)\Big|_e$$

$$= \left(\frac{d}{dt_k}\Big|_{t_k=0} \cdots \frac{d}{dt_1}\Big|_{t_1=0} \right) \left(f(\exp(t_k A + \cdots + t_1 A)) \right)$$

$$= \exp_*(A^{\otimes k})(f)$$

So

$$\exp_*(A^{\otimes k}) = (X^A)^k\big|_e = A \cdots A$$
$$= \mathrm{pbw}(A^{\otimes k})$$

Step 2:

exer:

A general symmetric tensor

$$X_1 \odot X_2 \odot \cdots \odot X_k \in S(\mathfrak{g})$$

can be expressed as a linear combination of elements of the form $A^{\odot k}$.

e.g.

$$X_1 \odot X_2 = \frac{\overset{X_1^{\odot 2} + 2X_1 \odot X_2 + X_2^{\odot 2}}{\overset{\shortparallel}{(X_1+X_2)^{\odot 2}}} - (X_1 - X_2)^{\odot 2}}{4}$$

$\Rightarrow$ Thm.

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