Geometric models for noncommutative algebras
Fall 2025, week 3

Left invariant vector fields

Recall:
If G is a matrix, i.e. G is a closed subgp of $GL_n(\mathbb{C})$, then

$\{ \gamma'(0) \mid \gamma : (-\varepsilon, \varepsilon) \to G,\ \gamma(0) = e \}$

$e = I_n$

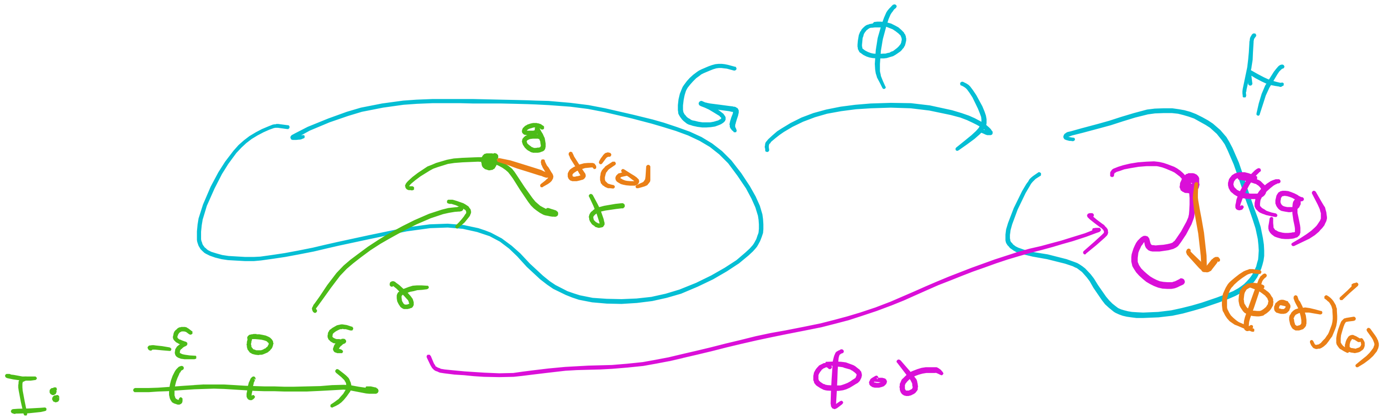
$$T_e G = \mathrm{Lie}(G) = \mathfrak{g},$$

with $[A, B] = AB - BA$, is a Lie algebra.

Today:
Another explanation of $\mathrm{Lie}(G)$

Let
$$\phi : G \to H$$
be a smooth map between matrix groups.
The tangent map of ϕ at $g \in G$ is the linear map
$$\phi_* = T_g \phi \,(= D_g \phi) : T_g G \to T_{\phi(g)} H$$

$$\phi_*(\gamma'(0)) = (\phi\circ\gamma)'(0)$$



For $g\in G$, consider

$$L_g : G \to G,\quad L_g(h) = gh$$

$$R_g : G \to G,\quad R_g(h) = hg$$

$\Rightarrow$ we have

$$T_h L_g : T_h G \to T_{\underbrace{L_g(h)}_{=gh}} G$$

$$T_h R_g : T_h G \to T_{\underbrace{R_g(h)}_{=hg}} G$$

and $\quad X$

and

$$(L_g)_* : \mathfrak{X}(G) \longrightarrow \mathfrak{X}(G),$$

$$((L_g)_* X)_h = (T_{g^{-1}h} L_g)(X_{g^{-1}h}) \in T_h G$$

$$(R_g)_* : \mathfrak{X}(G) \longrightarrow \mathfrak{X}(G),$$

$$((R_g)_* X)_h = (T_{hg^{-1}} R_g)(X_{hg^{-1}}) \in T_h G$$

<u>Def</u> (right invariant)

A vector field $X \in \mathfrak{X}(G)$ is <u>left invariant</u> if

$$\underbrace{(L_g)_* X = X}_{} \quad ((R_g)_* X = X) \quad \forall g \in G.$$

$$\Updownarrow$$

That is,

$$((L_g)_* X)_{\cancel{h}\, gh} = X_{\cancel{h}\, gh}$$

$$\shortparallel$$

$$(T_{\cancel{g^{-1}h}\, h} L_g)(X_{\cancel{g^{-1}h}\, h}) \quad \forall g, h \in G.$$

We denote

$$\mathfrak{X}^L(G) = \{\text{left invariant vec. fields on } G\}$$

$$\mathfrak{X}^R(G) = \{\text{right inv. vec. fields on } G\}$$

<u>Prop</u>

A vector field $X \in \mathfrak{X}(G)$ is left invariant iff

$$X(f \circ L_g) = X(f) \circ L_g$$

$\forall\, g \in G$, $f \in C^\infty(G)$.

Pf

"$\Rightarrow$" Let $X \in \mathfrak{X}^L(G)$. For each $g \in G$, choose $\gamma_g : I \to G$ s.t. $\gamma_g'(0) = X_g$

Since

$$X_{gh} = (T_h L_g)(X_h)$$

$$\gamma_{gh}'(0) \qquad (L_g \circ \gamma_h)'(0)$$

$$\Rightarrow (X(f)\circ L_g)(h) = X_{gh}(f) = \cancel{(f\circ\gamma_{gh})'(0)}$$

$$= (f\circ(L_g\circ\gamma_h))'(0)$$

$$= (X(f\circ L_g))(h) \qquad \forall h, g \in G \quad \forall f \in C^\infty(G)$$

$$= \quad X(f)\circ L_g = X(f\circ L_g)$$

" $\Leftarrow$ ": exercise $\#$

Prop

The space $\mathfrak{X}^L(G)$ of left inv. vec fields is Lie subalg. of $\mathfrak{X}(G)$.

Recall : $[X,Y] = X\circ Y - Y\circ X$

pf

$\forall X, Y \in \mathfrak{X}^L(G)$, claim: $[X,Y] = X\circ Y - Y\circ X \in \mathfrak{X}^L(G)$

check: $\forall f \in C^\infty(G)$ $\forall g \in G$,

$$[X,Y](f\circ L_g) \qquad \left(\overset{?}{=} [X,Y](f)\circ L_g\right)$$

$$= \Big(X\big(\underline{Y(f\circ L_g)}\big)\Big) - \Big(Y\big(\underline{X(f\circ L_g)}\big)\Big)$$

$X, Y \in \mathfrak{X}^l(G)$ $\qquad \| \; Y \in \mathfrak{X}^l(G) \qquad \| \; X \in \mathfrak{X}^l(G)$

$$= \Big(X\big(\underline{Y(f)\circ L_g}\big)\Big) - Y\big(\underline{X(f)\circ L_g}\big)$$

$$= \underline{X}\big(Y(f)\big)\circ L_g - \underline{Y\big(X(f)\big)}\circ L_g$$

$$= [X,Y](f)\circ L_g \qquad \#$$

Thm 1

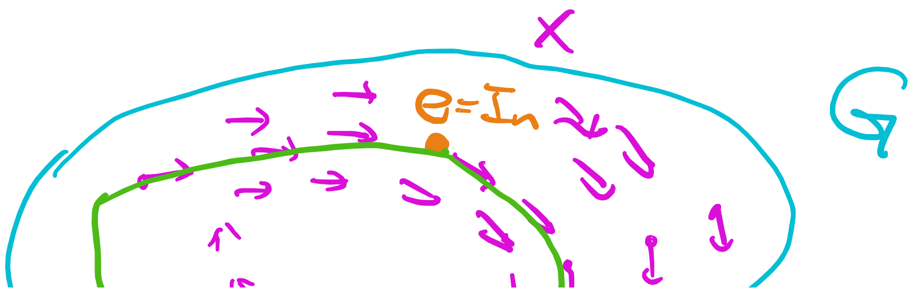
Let G be a matrix group.

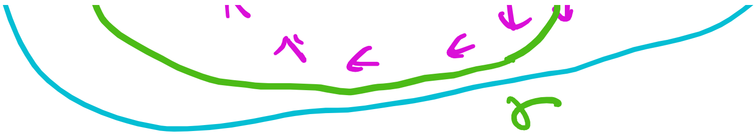
If $X \in \mathfrak{X}^l(G)$, then $\exists!\, \gamma : \mathbb{R} \to G$ s.t.

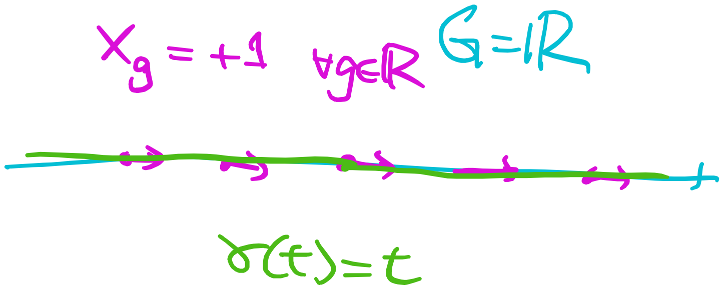
(i) $\gamma'(t) = X_{\gamma(t)} \quad \forall\, t \in \mathbb{R}$

(ii) $\gamma(0) = e$

(iii) $\gamma(s+t) = \gamma(s)\cdot\gamma(t) \quad \forall\, s,t \in \mathbb{R}$







pf

Let $G \subseteq GL_n(\mathbb{C})$ be a matrix gp.

and $A \in \mathfrak{g} = \mathrm{Lie}(G) = T_eG \subseteq M_n(\mathbb{C})$,

By the definition of $\mathfrak{g}$, $\exists\ \gamma_A : (-\varepsilon, \varepsilon) \to G$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{I}$

s.t. $\gamma_A(0) = e$, $\quad \gamma_A'(0) = A$

Given $g \in G$, since

$$M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad B \longmapsto g \cdot B$$

is a linear map, we have

$$(g \cdot \gamma)'(0) = g \cdot (\gamma'(0))$$

Also note that

$$g\cdot\gamma : (-\varepsilon, \varepsilon) \to G$$

is a curve s.t.

$$(g\cdot\gamma)(0) = g\cdot\gamma(0) = g\cdot e = g$$

$$(g\cdot\gamma)'(0) = g\cdot A \in T_g G$$

Consider

$$X^A_g := g\cdot A \in T_g G$$

$\Rightarrow$ $X^A \in \mathfrak{X}(G)$, and

$$(T_{g^{-1}h} L_g)(X^A_{g^{-1}h}) = g\cdot(g^{-1}h\cdot A)$$
$$= h\cdot A = X^A_h$$

So $\begin{cases} X^A \in \mathfrak{X}^L(G) \text{ s.t.} \\ X^A_e = A \end{cases}$

Note that a left-inv v.f. $X \in \mathfrak{X}^L(G)$ is uniquely determined by $X_e \in T_e G$:

$$(T_{e} L_g)(X_{e}) = X_g$$

$A = X_e$
$\rightsquigarrow X^A$
$X^A_g = (T_e L_g)(X^A_e) = T_e L_g(X_e) = X_g$

So any left inv. v.f. on G is of the form X^A for some $A \in \mathfrak{g}$.

By $\exists!$ Thm of IVP, $\exists\, \delta > 0$ and $\exists\, \gamma : (-\delta, \delta) \to G$ s.t.

$$\begin{cases} \gamma'(t) = X^A_{\gamma(t)} = \gamma(t) \cdot A \\ \gamma(0) = e \end{cases}$$

<u>Claim:</u> δ can be chosen to be ∞

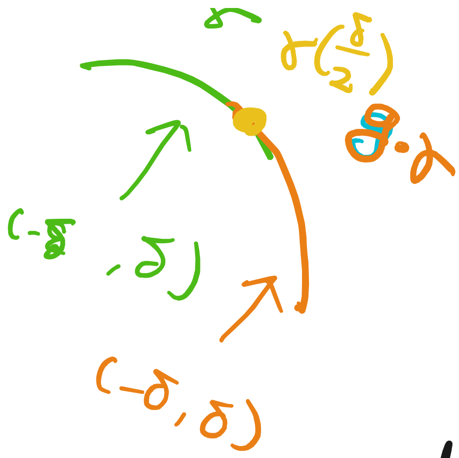
<u>pf of Claim</u>

For $g \in G$,

$$(g \cdot \gamma)'(t) = g \cdot \gamma'(t) = g \cdot X^A_{\gamma(t)}$$
$$= g \cdot \gamma(t) \cdot A = X^A_{g \cdot \gamma(t)}$$

$\Rightarrow$ $g \cdot \gamma : (-\delta, \delta) \to G$ is a sol. of

$$\begin{cases} \alpha'(t) = X^A_{\alpha(t)} \\ \alpha(0) = g \end{cases}$$



Taking $g = \gamma(\frac{\delta}{2})$, we can extend

$$\gamma : (-\delta, \delta) \to G$$

to an integral curve

$$\gamma : (-\delta, \tfrac{3}{2}\delta) \to G$$

Inductively, we can extend π to

$$\gamma : (-\infty, \infty) \to G.$$

s.t.

$$\begin{cases} \gamma'(t) = X^A_{\gamma(t)} \\ \gamma(0) = e \end{cases}$$

Furthermore, since

$$\begin{cases} \left.\frac{d}{dt}\right|_{t=0} \gamma(s+t) = \gamma'(s) = X^A_{\gamma(s)} \end{cases}$$

($\gamma(s+t)\big|_{t=0} = \gamma(s)$)

we have

by the uniquess of sol.

$$\gamma(s+t) = \gamma(s)\cdot\gamma(t) \qquad \#$$

One-parameter subgp and exp

The previous thm leads to the following def:

Def

A one-parameter subgroup of G is is a smooth gp homo of the form

$$\gamma : \mathbb{R} \to G$$
$$\gamma(s+t) = \gamma(s)\cdot\gamma(t)$$
$$\gamma(0) = e$$

To describe γ, recall

① $\exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A}{n!} = e^A$$

② $$\frac{d}{dt} e^{tA} \overset{\text{exer}}{=} \sum_{n=0}^{\infty} \left(\frac{t^n \cdot A^n}{n!} \right)'$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{t^{n-1} \cdot A^{n-1}}{(n-1)!}}_{= e^{tA}} \cdot A$$

$$= e^{tA} \cdot A = A \cdot e^{tA}$$

$(sA)\cdot(tA) = (tA)\cdot(sA)$

③ $$e^{(s+t)A} = e^{sA} \cdot e^{tA}$$

$\Rightarrow$ $\gamma(t) = e^{tA}$ is a one-parameter subgp

Lemma

Let $A \in M_n(\mathbb{C})$. If $y : (a,b) \to M_{1\times n}(\mathbb{C})$ s.t.

$$y' = y \cdot A \quad (*)$$

$n\times 1$

then

$$y(t) = C \cdot e^{tA}$$

for some $C \in M_{1\times n}(\mathbb{C})$.

$n \times 1$

pf $\quad e^{-tA} \cdot y$

$$\circledast \Rightarrow \left(y \cdot e^{-tA}\right)' \qquad -A \cdot e^{-tA}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel \text{ computation}$$

$$= \underbrace{y'}_{\substack{\parallel \circledast \\ y \cdot A}} \cdot e^{-tA} + y \cdot \underbrace{\left(e^{-tA}\right)'}$$

$$= y \cdot A e^{-tA} - y \cdot A \cdot e^{-tA} = 0$$

$$\Rightarrow \quad y \cdot e^{-tA} = C \quad \text{for some } C \in M_{1\times n}(\mathbb{C}).$$

$$\Rightarrow \quad y = C \cdot e^{tA} \qquad \#$$

Thm2

$\gamma : \mathbb{R} \to GL_n(\mathbb{C})$ is a one-parameter subgp

$\Longleftrightarrow$ $\exists!\ A \in M_{n\times n}(\mathbb{C})$ s.t.

$$\gamma(t) = \exp(tA) = e^{tA}.$$

pf

" $\Leftarrow$ " is done.

"$\Rightarrow$" Assume $\gamma: \mathbb{R} \to GL_n(\mathbb{C})$ is a one-parameter subgp

Set

$$A := \gamma'(0) \in M_n(\mathbb{C}).$$

By assumption, $\forall s,t \in \mathbb{R}$

$$\gamma(s+t) = \gamma(s) \cdot \gamma(t)$$

$\left.\frac{d}{ds}\right|_{s=0} \Rightarrow$

$$\underline{\gamma'}(t) = \gamma'(0) \cdot \gamma(t) \underline{= A \cdot \gamma(t)}$$

Lemma $\Rightarrow$

$$\gamma(t) = \exp(tA) \cdot C$$

Since

$$\gamma(0) = I_n = \exp(0 \cdot A) \cdot C = C$$

we have

$$\gamma(t) = \exp(tA).$$

#

<u>Cor</u>

For a matrix gp $G \subseteq GL_n(\mathbb{C})$,

$$\exp(\mathfrak{g}) \subseteq G.$$

pf

Let $A \in \mathfrak{g} = T_e G$.

Consider the left invariant vec field

$$X^A \in \mathfrak{X}^L(G), \quad X^A_g = g \cdot A$$

By Thm 1, the integral curve

$$\gamma : \mathbb{R} \to G, \quad \gamma'(t) = X^A_{\gamma(t)}, \quad \gamma(0) = e$$

defines a one-parameter subgp.

Since $\gamma'(0) = A$, by Thm 2,

$$\gamma(t) = \exp(tA)$$

$$\Rightarrow \exp(A) = \gamma(1) \in G. \qquad \#$$

$Lie(G)$ & $\mathfrak{X}^L(G)$

Thm

Let G be a matrix gp. The map

$$\mathfrak{X}^L(G) \longrightarrow \mathfrak{g} = T_e G$$

$$\begin{matrix} \Psi & & \Psi \end{matrix}$$

$$X \longmapsto X_e$$

is an isomorphism of Lie algebras.

pf

<u>1-1</u>:

$X \in \mathfrak{X}^L(G)$ is uniquely determined by $\underset{\in}{X_e}^{\mathfrak{g}}$

<u>onto</u>:

Given $A \in \mathfrak{g}$, we have $X^A \in \mathfrak{X}^L(G)$

with $X^A_e = A$.

<u>Compatibility with Lie brackets</u>:

For $X, Y \in \mathfrak{X}^L(G)$, we need to show

$$\underset{(X\circ Y - Y \circ X)_e}{\underset{\shortparallel}{[X,Y]_e}} = [X_e, Y_e] = X_e \cdot Y_e - Y_e \cdot X_e$$

matrix multiplication

Note that

$$\begin{array}{ccc} \mathfrak{X}^L(G) & \longleftrightarrow & \mathfrak{g} \\ & & \rotatebox{90}{$\in$} \\ X^A & \longleftarrow & A \end{array}$$

$X^A_g = g \cdot A$

Let $A, B \in \mathfrak{g}$. $\rightsquigarrow$ $X^A, X^B \in \mathfrak{X}^L(G)$.

Let Φ^A, Φ^B be the flows of X^A and X^B

respectively $\quad \phi_t^A : G \to G,\ \forall t \in \mathbb{R},\quad \frac{d}{dt}\phi_t^A(g) = X^A_{\phi_t^A(g)}$

By Thm 1 and Thm 2,

$$\phi_t^A : G \to G, \quad \phi_t^A(g) = g \cdot \exp(tA)$$

$$\phi_t^B : G \to G, \quad \phi_t^B(g) = g \cdot \exp(tB).$$

$\forall t \in \mathbb{R}.\ g \in G.$

By Thm from last week

$$\Rightarrow [X^A, X^B]_e = \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}\Big(\phi_{-t}^B \circ \phi_{-t}^A \circ \underbrace{\phi_t^B \circ \underbrace{\phi_t^A(e)}_{=\, e\cdot\exp(tA) \,=\, \exp(tA)}}_{=\,\phi_t^B(\exp(tA)) \,=\, \exp(tA)\cdot\exp(tB)}\Big)$$

$$= \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}\Big(\underbrace{\exp(tA)}_{=\sum \frac{t^n A^n}{n!}} \cdot \underbrace{\exp(tB)}_{=\sum \frac{t^n B^n}{n!}} \cdot \underbrace{\exp(-tA)}_{=\sum \frac{(-t)^n A^n}{n!}} \cdot \underbrace{\exp(-tB)}_{=\sum \frac{(-t)^n B^n}{n!}}\Big)$$

$$= \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}\Bigg(I_n + t\underbrace{(A + B - A - B)}_{=0} + \boxed{t^2\Big(\cancel{\frac{A^2}{2}} + \cancel{\frac{B^2}{2}} + \cancel{\frac{A^2}{2}} + \cancel{\frac{B^2}{2}} + \cancel{AB} - \cancel{A^2} - \cancel{AB} - BA - \cancel{B^2} + AB\Big)} + O(t^3)\Bigg)$$

$$= \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}\Big(t^2(AB - BA)\Big)$$

$= AB - BA = [A, B]$ #

Summary:

Let $G \subseteq Gl_n(\mathbb{C})$ be a matrix gp.

Then

① $X \in \mathfrak{X}^L(G) \iff X_g = g \cdot A$ for some $A \in \mathfrak{g}$, where $X_g = X^A_g$

② Φ^A_t is a flow of X^A

$\iff \Phi^A_t(t) = g \cdot \exp(tA)$

③ $\exp : \mathfrak{g} \to G$, $A \mapsto \sum_{n=1}^{\infty} \frac{A^n}{n!}$

④ $\mathfrak{X}^L(G) \to \mathfrak{g}$, $X \mapsto X_e$,

is a Lie alg iso.

Next (see youtuble list).

Consider

$$\mathcal{U}\mathfrak{g} = \mathcal{T}\mathfrak{g} / \langle x \otimes y - y \otimes x - [x, y] \rangle$$

Note: for $X, Y \in \mathfrak{g} \subseteq \mathcal{U}\mathfrak{g}$,

$$X \cdot Y = Y \cdot X + [X, Y]$$

We want to understood $\mathcal{U}\mathfrak{g}$ as left invariant differential operators on G if $\mathfrak{g} = \mathrm{Lie}(G)$.