Geometric models for noncommutative algebras
Fall 2025, week 2

Recall

A matrix group is a closed subgp of $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) \neq 0\}$

$$\subset_{\text{open}} M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$$

Example

① $$GL_n(\mathbb{R}) = \underbrace{M_n(\mathbb{R})}_{\cong\, \mathbb{R}^{n^2} \subseteq \mathbb{R}^{2n^2}} \cap GL_n(\mathbb{C})$$

$$= \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$$

general linear group (over $\mathbb{R}$)

② The special linear group (over $\mathbb{C}$) is

$$SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) \mid \det(A) = 1\}$$

$$= \underbrace{\det^{-1}(\{1\})}_{\substack{\cap \text{ closed} \\ M_n(\mathbb{C})}} \cap GL_n(\mathbb{C})$$

③ The unitary group (of degree n) is

$$U(n) = \{A \in GL_n(\mathbb{C}) \mid A \cdot A^* = A^* \cdot A = I_n\},$$

where

$$A^* = (\bar{A})^T$$

Note:

(a) $U(n) =$ sol. set of

$$\left.\underbrace{A^{-1} - A^*}_{\substack{\uparrow \\ \text{it is a continuous function } \Phi \\ GL_n(\mathbb{C}) \to M_n(\mathbb{C})}} = 0 \right\} = \Phi^{-1}(\{0\})$$

$\underset{\text{closed}}{\subset} GL_n(\mathbb{C})$

(b) $n=1$:

$$U(1) = \{ \overset{z\in\mathbb{C}}{A} \in GL_1(\mathbb{C}) \mid A\overset{\bar{z}}{A^*} = \overset{(1)}{I_1} \}$$

$$= \{ z \in \mathbb{C} \mid \underset{a^2+b^2}{\overset{a+bi}{z}\,\bar{z}} = 1 \}$$

$$\cong S^1 = \{ (a,b) \in \mathbb{R}^2 \mid a^2+b^2 = 1 \}$$

$=$ unit circle in $\mathbb{C} \cong \mathbb{R}^2$

(c) The special unitary group of deg n is

$$SU(n) = U(n) \cap SL_n(\mathbb{C})$$

④ The orthogonal group in dim. n is

$$O(n) = U(n) \cap GL_n(\mathbb{R})$$

$$= \{ A \in M_n(\mathbb{R}) \mid A \cdot A^T = A^T A = I_n \}$$

$=$ the group of distince-preserving linear transformations in $\mathbb{R}^n$.

exercise

NOTE:

(a) $n=2$:

exercise

$$O(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$$

(b) The special orthogonal group in dim n

$$SO(n) = O(n) \cap SL_n(\mathbb{R})$$
$$= \left\{ A \in M_n(\mathbb{R}) \,\middle|\, \begin{array}{l} A \cdot A^T = A^T \cdot A = I_n, \\ \det(A) = 1 \end{array} \right\}$$

e.g.

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$$

Lie algebras of matrix groups

Let $I = (-\varepsilon, \varepsilon)$ be an open interval, and

$$\gamma : I \longrightarrow M_{m\times n}(\mathbb{C}).$$

We say γ is differentiable (resp. smooth, continuous) if as a map

$$\mathbb{R} \underset{\text{open}}{\supset} I \to M_{m\times n}(\mathbb{C}) \cong \mathbb{C}^{mn} \cong \mathbb{R}^{2mn}$$

it is differentiable (resp. smooth, continuous)

More explicitly,

$$\gamma(t) = \begin{pmatrix} a_{11}(t) + i\, b_{11}(t) & \cdots & a_{1n}(t) + i\, b_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{m1}(t) + i\, b_{m1}(t) & \cdots & a_{mn}(t) + i\, b_{mn}(t) \end{pmatrix}$$

is differentiable (resp. smooth, continuous) if all $a_{jk}(t)$ and $b_{jk}(t)$, $\forall j,k$, are differentiable (resp. smooth, continuous)

If $\gamma(t)$ is differentiable, then

$$\gamma'(t) = \begin{pmatrix} a_{11}'(t)+i\,b_{11}'(t) & \cdots & a_{1n}'(t)+i\,b_{1n}'(t) \\ \vdots & \ddots & \vdots \\ a_{m1}'(t)+i\,b_{m1}'(t) & \cdots & a_{mn}'(t)+i\,b_{mn}'(t) \end{pmatrix}$$

Pf: exercise

Prop

Let $\alpha : I \to M_{m\times n}(\mathbb{C})$, $\beta : I \to M_{n\times p}(\mathbb{C})$, $\gamma : I \to M_n(\mathbb{C})$ be differentiable. Then

(i) $$\big(\overbrace{\alpha(t)\cdot\beta(t)}^{m\times p \text{ matrix}}\big)' = \alpha'(t)\cdot\beta(t) + \alpha(t)\cdot\beta'(t)$$

(ii) If $\gamma(t)$, $t\in I$, are invertible and $\gamma(0) = I_n$, then

$$\frac{d}{dt}\Big|_{t=0}\left(\gamma'(t)\right) = -\gamma'(0)$$

(iii) If

$$\gamma(t) = \begin{pmatrix} z_{11}(t) & \cdots & z_{1n}(t) \\ \vdots & \ddots & \vdots \\ z_{n1}(t) & \cdots & z_{nn}(t) \end{pmatrix}$$

then

$$\left(\det(\gamma(t))\right)'$$

$$= \det\begin{pmatrix} z_{11}'(t) & z_{12}(t) & \cdots & z_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1}'(t) & z_{n2}(t) & \cdots & z_{nn}(t) \end{pmatrix}$$

$$+ \det\begin{pmatrix} z_{11}(t) & z_{12}'(t) & \\ \vdots & \vdots & \cdots \\ z_{n1}(t) & z_{n2}'(t) & \end{pmatrix}$$

$$+ \cdots$$

$$+ \det\begin{pmatrix} z_{11}(t) & \cdots & z_{1(n-1)}(t) & z_{1n}'(t) \\ \vdots & \ddots & \vdots & \vdots \end{pmatrix}$$

$$\left. z_{n1}(t) \quad \cdots \quad z_{n(n-1)}(t) \quad z_{nn}'(t) \right|$$

e.g. $$\left(\begin{vmatrix} a(t) & b(t) \\ c(t) & d(t) \end{vmatrix} \right)' = \begin{vmatrix} a'(t) & b(t) \\ c'(t) & d(t) \end{vmatrix} + \begin{vmatrix} a(t) & b'(t) \\ c(t) & d'(t) \end{vmatrix}$$

Def

Let G be a matrix group.

The Lie algebra associated with G is

$$\mathfrak{g} = \mathrm{Lie}(G) = T_e G \quad (e = \text{identity matrix})$$

$$= \left\{ \gamma'(0) \in M_n(\mathbb{C}) \;\middle|\; \begin{array}{l} \gamma : \underset{(-\varepsilon, \varepsilon)}{I} \to G \\ \text{is smooth,} \\ \gamma(0) = e = I_n \end{array} \right\}$$

Lemma

$\mathfrak{g} = \mathrm{Lie}(G)$ is a real vector subspace of $M_n(\mathbb{C})$

pf

① $0 \in \mathfrak{g}$, since

$$0 = (\underset{\substack{\| \\ \gamma(t)}}{\text{constant curve}})'\Big|_{t=0} \in \mathfrak{g}$$

$\gamma(t) = I_n \quad \forall t \in I$.

② For $\alpha'(0)$, $\beta'(0) \in \mathfrak{g}$, $\left(\begin{array}{l}\alpha, \beta : I \to G \\ \alpha(0) = \beta(0) = I_n\end{array}\right)$.

$$(\alpha \cdot \beta)'(0) = \alpha'(0) \cdot \underbrace{\beta(0)}_{\overset{\|}{I_n}} + \underbrace{\alpha(0)}_{\overset{\|}{I_n}} \cdot \beta'(0)$$

$$= \alpha'(0) + \beta'(0) \in \mathfrak{g}$$

③ For $a \in \mathbb{R}$, $\gamma'(0) \in \mathfrak{g}$,

$$\big(\gamma(a \cdot t)\big)'\Big|_{t=0} = \gamma'(0) \cdot (at)'$$

$$= a \cdot \gamma'(0) \in \mathfrak{g}$$

#

Thm

$\mathfrak{g} = \mathrm{Lie}(G)$ is a real Lie subalgebra

$\subseteq M_n(\mathbb{C})$ $([A, B] = AB - BA)$

of $M_n(\mathbb{C})$ $([A,B] = AB - BA)$

pf

It suffices to show: $\mathfrak{g} = \mathrm{Lie}(G)$ is closed under $[A,B] = AB - BA$.

Let $A = \alpha'(0)$, $B = \beta'(0) \in \mathfrak{g}$.

$\alpha, \beta : I \to G$

Consider

$$F : I \times I \to G,$$

$$F(s,t) := \alpha(s) \cdot \beta(t) \cdot (\alpha(s))^{-1} \in G$$

Define

$$\gamma : I \to \mathfrak{g}, \qquad \mathfrak{g} \subseteq G$$

$$\gamma(s) := \left.\frac{d}{dt}\right|_{t=0} F(s,t) = \alpha(s) \cdot \beta'(0) \cdot (\alpha(s))^{-1}$$

Since $\mathfrak{g}$ is a real vector subspace of $M_n(\mathbb{C})$

we have

$$\lim_{s \to 0} \frac{\gamma(s) - \gamma(0)}{s} \in \mathfrak{g}$$

''

$$\gamma'(0) = \alpha'(0)\cdot\beta'(0)\cdot(\overbrace{\alpha(0)}^{I_n})^{-1} + \overbrace{\alpha(0)}^{I_n}\cdot\beta'(0)\,\underbrace{\frac{d}{ds}\Big|_{s=0}(\alpha(s))^{-1}}_{=-\alpha'(0)}$$

$$= \alpha'(0)\cdot\beta'(0) - \beta'(0)\cdot\alpha'(0) \in \mathfrak{g}$$

$$= [A,B]$$

To compute Lie(G), it's useful to have the exponential map:

<u>Thm</u>

The series

$$\exp(A) = e^A = \sum_{k=0}^{\infty}\frac{1}{k!}A^k$$

defines a (smooth) map

$$M_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$$

s.t.

$\exp(A+B) = \exp(A) \ldots$

$$\exp(A+B) = \exp(A)\cdot\exp(B)$$

whenever $AB = BA$.

This map $\exp : M_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is called the exponential map.

Remark

① $$\frac{d}{dt}\exp(t\cdot A) = \exp(tA)\cdot A = A\cdot\exp(tA)$$

② So $\gamma(t) = \exp(tA) : I \to GL_n(\mathbb{C})$ is smooth, $\gamma(0) = \exp(0) = \sum_{k=0}^{\infty}\frac{1}{k!}0^k = I_n$

Example

① $$\mathrm{Lie}(GL_n(\mathbb{C})) = \mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C})$$

pf

(a) $\mathfrak{gl}_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ by Thm

(b) $\mathfrak{gl}_n(\mathbb{C}) \supseteq M_n(\mathbb{C})$:

$\forall A \in M_n(\mathbb{C})$ consider

$\forall A \in M_n(\mathbb{C})$, consider

$\gamma(t) = \exp(tA) : \mathbb{R} \to GL_n(\mathbb{C})$

smooth, $\gamma(0) = I_n$, $\gamma'(0) = A \cdot \exp(tA)|_{t=0}$

$= A \quad \in \mathfrak{gl}_n(\mathbb{C})$ #

② Similarly, $Lie(GL_n(\mathbb{R})) = \mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$

③ $Lie(\underbrace{SL_n(\mathbb{C})}_{}) = \mathfrak{sl}_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid tr(A) = 0\}$

$SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) = 1\}$

pf

"$\supseteq$" $\forall A \in M_n(\mathbb{C})$, $tr(A) = 0$. consider

$\gamma(t) = \exp(tA)$

exercise

lemma

$\Rightarrow \quad det(\gamma(t)) = det(\exp(tA)) = \exp(tr(tA))$

$= \exp(0) = e^0 = 1$

"$\subseteq$" $\gamma : I \to M_n(\mathbb{C})$, $det(\gamma(t)) = 1$

exer

$\Rightarrow \quad 0 = \frac{d}{dt}\Big|_{t=0} det(\gamma(t)) = \cdots\cdots$

$= tr(\gamma'(0))$

$\Rightarrow \gamma'(0) \in \mathfrak{sl}_n(\mathbb{C})$ #

④ $\mathrm{Lie}(SL_n(\mathbb{R})) = \mathfrak{sl}_n(\mathbb{R}) \overset{\exp}{=} \{A \in M_n(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$

⑤ $\mathrm{Lie}(U(n)) = \mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) \mid A + A^* = 0\}$

pf

"$\subseteq$" $\gamma: I \to U(n) = \{A \in M_n(\mathbb{C}) \mid A \cdot A^* = I_n\}$

$$\gamma(t) \cdot \gamma(t)^* = I_n$$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} (\gamma(t) \cdot \gamma(t)^*) = 0$$

$$\| \qquad I_n \qquad\qquad I_n$$

$$\gamma'(0) \cdot \underbrace{\gamma(0)^*}_{} + \underbrace{\gamma(0)}_{} \cdot (\gamma'(0)^*)$$

$$\gamma'(0) + (\gamma'(0))^* = 0$$

$\Rightarrow A^* = -A \Rightarrow A^* \cdot A = A \cdot A^*$

"$\supseteq$" $\forall A \in M_n(\mathbb{C})$, $A + A^* = 0$, Consider

$$\gamma(t) = \exp(t \cdot A). \quad \Rightarrow \gamma(0) = I_n$$

$$\Rightarrow \quad \gamma(t)^* \cdot \gamma(t) = \exp(tA^*) \cdot \exp(tA)$$

$$= \exp(tA^* + tA) = \exp(0) = I.$$

$\Rightarrow \gamma(t) \in U(n) \Rightarrow \gamma'(0) \in \mathfrak{u}(n)$

⑥ $\mathrm{Lie}(SU(n)) = \mathfrak{su}(n) = \left\{ A \in M_n(\mathbb{C}) \;\middle|\; \begin{array}{l} A + A^* = 0 \\ \mathrm{tr}(A) = 0 \end{array} \right\}$

⑦ $\mathrm{Lie}(O(n)) = \mathfrak{o}(n) \overset{\text{exer}}{=} \{ A \in M_n(\mathbb{R}) \mid A + A^T = 0 \}$

⑧ $\mathrm{Lie}(SO(n)) = \mathfrak{so}(n) = \mathfrak{o}(n)$

Lie algebras as left invariant vec fields

Let G be a matrix group. $g \in G$.

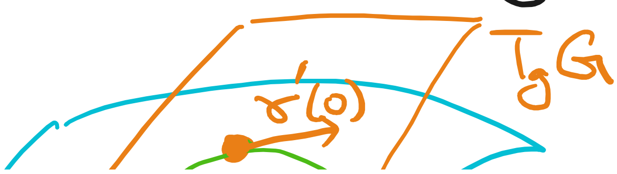
Recall

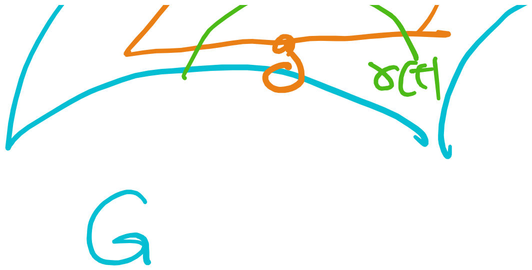
$$T_g G = \underline{\text{tangent space}} \text{ of } G \text{ at } g$$

$$= \left\{ \gamma'(0) \in M_n(\mathbb{C}) \;\middle|\; \begin{array}{l} \gamma : (-\varepsilon, \varepsilon) \to \underset{\text{II}}{\overset{I}{G}}, \\ \gamma(0) = g \end{array} \right\}$$

In particular,

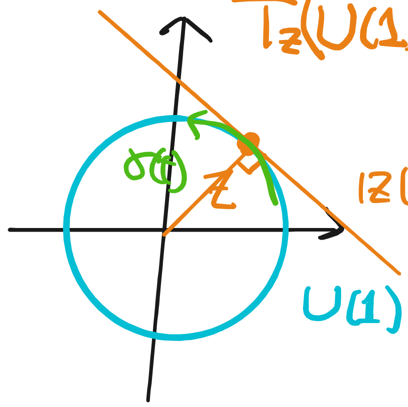
$\mathrm{Lie}(G) = \mathfrak{g} =$ tangent space of G at $e = I_n$





$z = a+bi$

e.g. $G = U(1)$



$T_z(U(1)) = \{x+iy \in \mathbb{C} \mid ax+by=0\}$

$|z| = 1$

$\gamma(t) = \alpha(t) + i\cdot\beta(t)$

$\alpha(t)^2 + \beta(t)^2 = 1$

$\frac{d}{dt}\Big|_{t=0}$

$\gamma(0) = \alpha(0) + i\beta(0) = z$

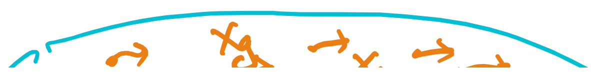
$= a + ib$

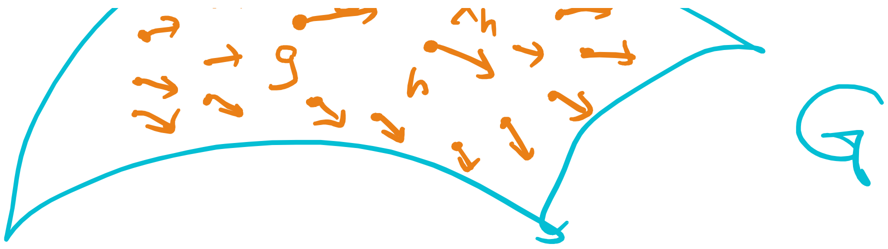
$$2\alpha(0)\cdot\alpha'(0) + 2\beta(0)\cdot\beta'(0) = 0$$

$$\| $$

$$2\langle (a,b), (\alpha'(0), \beta'(0))\rangle = 0$$

A vector field X on G is a smooth map $X: G \to G \times M_n(\mathbb{C})$, $g \mapsto (g, X_g)$ s.t. $X_g \in T_g G \quad \forall g \in G$





$\mathfrak{X}(G) :=$ the set of vector fields on G

<u>Remark</u>

A derivation X of an algebra A is a linear map $X : A \to A$ s.t.

$$X(a \cdot b) = X(a) \cdot b + a \cdot X(b)$$

The set of derivations of A is denoted by $\mathrm{Der}(A)$.

Consider $A = C^\infty(G)$

$GL_n(\mathbb{C}) \subseteq M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$

$$\overset{\cup\!\!\mid}{} = \left\{ f : G \to \mathbb{R} \;\middle|\; f = F|_G \text{ for some smooth function } F : M_n(\mathbb{C}) \to \mathbb{R} \right\}$$

<u>Thm</u>

There is a natural identification:

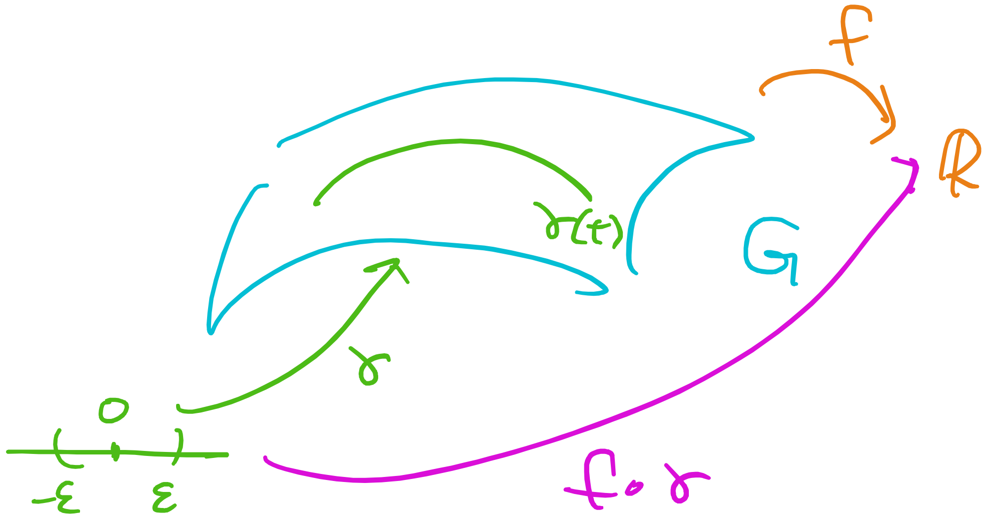
$$\Phi : \mathfrak{X}(G) \longleftrightarrow \mathrm{Der}(C^\infty(G))$$

$$\underset{X}{\Psi} \longmapsto \underset{\Phi(X)}{\Psi}$$

$f \in C^\infty(G)$

$f : G \to \mathbb{R}$

$$\Big(\underbrace{(\Phi(X))(f)}_{\in\, C^\infty(G)}\Big)\underbrace{(g)}_{\in\, G} = \underbrace{X_g}_{=\,\gamma'(0)}(f) := (f\circ\gamma)'(0) \in \mathbb{R}$$



$f_1, f_2 \in C^\infty(G)$

$$\left\{ Y : C^\infty(G) \xrightarrow{\text{linear}} \mathbb{R} \;\middle|\; Y(f_1 \cdot f_2) = \underbrace{Y(f_1)}_{\in\,\mathbb{R}} \cdot f_2(g) + f_1(g)\cdot Y(f_2) \right\} = \mathrm{Der}_g(C^\infty(G))$$

<u>Thm</u>

$$\Psi : T_gG \overset{\cong}{\longleftrightarrow} \mathrm{Der}_g(C^\infty(G))$$

$$\gamma'(0) = \underset{X_g}{\Psi} \longrightarrow \underset{\Psi(X_g)}{\Psi}$$

$$(\Psi(X_g))(f) = (f\circ\gamma)'(0)$$

Prop

$\mathrm{Der}(A)$ is a Lie algebra with $[X,Y]=X\circ Y-Y\circ X$

In particular, $\mathfrak{X}(G)$ is a Lie algebra

as a vector space, it is mostly infinite-dimensional.

The bracket $[X,Y]=X\circ Y-Y\circ X$ on $\mathfrak{X}(G)$ can be understood geometrically by flows:

Recall that a curve $\gamma:(-\varepsilon,\varepsilon)\to G$ is called an <u>integral curve</u> of a vector field $X\in\mathfrak{X}(G)$ if

$$\gamma'(t)=X_{\gamma(t)} \qquad \forall t\in(-\varepsilon,\varepsilon).$$

A flow of X is a smooth map

$$\phi : \underset{\underset{\mathbb{R}\times G \ni (t,g)}{\cap \text{ open}}}{U} \longrightarrow G, \qquad (t,g) \mapsto \phi_t(g)$$

defined on an open neighborhood U of $\{0\} \times G \subseteq \mathbb{R}\times G$, s.t.

(i) $\phi_0(g) = g \quad \forall g \in G$

(ii) $\left.\frac{d}{dt}\right|_{t=0} (\phi_t(g)) = X_p$

(iii) $\phi_{t+s}(g) = \phi_t(\phi_s(g))$ whenever both sides are defined

Thm (Existence, uniqueness and smooth dependence of solutions for an initial value problem)

Let $X \in \mathfrak{X}(G)$ and $g \in G$. $\exists\ \varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to G$ s.t.

$$\gamma'(t) = X_{\gamma(t)} \qquad \forall t \in (-\varepsilon, \varepsilon)$$

Ex

$\gamma(0) = g$

Furthermore, the integral curves fit together smoothly to define the unique local flow $\phi_t \in X$. $\left(\phi_t(g) = \gamma(t),\ \gamma(t) \text{ satisfying } (*) \right)$

<u>Thm</u>

Let ϕ_t^X and ϕ_t^Y be the local flows of vector fields X and Y, respectively.

Then $\forall g \in G$,

$$[X,Y]_g = \frac{1}{2} \frac{d^2}{dt^2}\Big|_{t=0} \left(\phi_{-t}^Y \circ \phi_{-t}^X \circ \phi_t^Y \circ \phi_t^X (g) \right)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\underline{\phi_{-\sqrt{t}}^Y \circ \phi_{-\sqrt{t}}^X \circ \phi_{\sqrt{t}}^Y \circ \phi_{\sqrt{t}}^X (g)} \right)$$