Geometric models for noncommutative algebras
Fall 2025, week 14

BV algebras and Lie algebroid connections

Recall:

$\exists!$ bilinear operation, called the Schouten bracket,

$$[\ ,\ ] : \Gamma(\Lambda^a T_M) \times \Gamma(\Lambda^b T_M) \to \Gamma(\Lambda^{a+b-1} T_M)$$

satisfying

(i) $[A,B] = -(-1)^{(a-1)(b-1)}[B,A]$

(ii) $[A, B\wedge C] = [A,B]\wedge C + (-1)^{(a-1)b} B\wedge[A,C]$

(iii) $(-1)^{(a-1)(c-1)}[A,[B,C]] + (-1)^{(b-1)(a-1)}[B,[C,A]]$
$+ (-1)^{(c-1)(b-1)}[C,[A,B]] = 0$

$A \in \Gamma(\Lambda^a T_M)$ A
$B \in \Gamma(\Lambda^b T_M)$ A, $C \in \Gamma(\Lambda^c T_M)$ A
$\forall f, g \in C^\infty(M)$

and

(iv) $[f,g] = 0$

(v) $[X, f] = X(f)$ $\quad \rho_X(f)$

$X, Y \in \mathfrak{X}(M)$ $\quad \Gamma(A)$

(vi) $[X,Y] = [X,Y]$.

$[X, Y] \quad [X, Y]_{\mathrm{Lie}}$

$[X, Y]_A$

Let $(A \to M, \rho)$ be a Lie algebroid.

$\exists !$ bilinear operation, called Schouten bracket,

$$[\ ,\ ] : \Gamma(\wedge^a A) \times \Gamma(\wedge^b A) \to \Gamma(\wedge^{a+b-1} A)$$

satisfying a version of (i) ~ (vi)

In this way, $(\Gamma(\wedge^\bullet A), \wedge, [\ ,\ ])$ forms a Gerstenhaber algebra.

<u>Def</u>

An operator D of degree -1 is called a generating operator of a Gerstenhaber algebra if

$$[a, b] = (-1)^{|a|}\Big(D(a \cdot b) - (Da) \cdot b - (-1)^{|a|} a \cdot (Db)\Big)$$

A Batalin-Vilkovisky algebra (or BV algebra) is a Gerstenhaber alg with a generating is characterized alg with a generating operator D s.t. $D^2 = D \circ D = 0$.

To see the relations between BV alg and Lie algebroids, we consider connections:

<u>Def</u>

Let $(A \to M, \rho)$ be a Lie algebroid,

$E \to M$ be a vector bundle.

An A-connection on E is an $\mathbb{R}$-bilinear map

$$\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E), \quad (X, S) \mapsto \nabla_X S$$

satisfying

$$\nabla_{fX} S = f \nabla_X S$$

$$\nabla_X (fS) = \rho_X(f) \cdot S + f \nabla_X S$$

for $f \in C^\infty(M)$.

The curvature R^∇ of an A-connection is

$[\nabla_X, \nabla_Y]$

exercise : D_{∇} ...

$$R^{\nabla}(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \Gamma(\mathrm{End}\,E)$$

$R(X,Y)(fs) = f R(X,Y)(s)$

An A-connection ∇ is flat if $R^{\nabla} = 0$

$R^{\nabla}(X,Y)S = 0$
$\forall X, Y \in \Gamma(A),\ S \in \Gamma(E)$

In the case $E = A$, the torsion of ∇ is

$$T^{\nabla} : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$$

$$T^{\nabla}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

Thm

Let $A \to M$ be a vector bundle.

(i) $\left\{ \begin{array}{l} \text{Gerstenhaber alg with} \\ \text{underlying graded alg} = (\Gamma(\wedge^{\bullet} A), \wedge) \end{array} \right\}$

$$\Big\updownarrow \text{1-1 onto}$$

$\left\{ \begin{array}{l} \text{Lie algebroid with underlying} \\ \text{vector bundle} = A \end{array} \right\}$

(ii) $\left\{ \begin{array}{l} \text{BV alg with underlying} \\ \text{graded alg} = (\Gamma(\Lambda^\bullet A), \wedge) \end{array} \right\}$

$$\updownarrow \text{ 1-1 onto}$$

$\left\{ \begin{array}{l} \text{Lie algebroids } A \text{ with a flat } A\text{-connection} \\ \text{on } \Lambda^{\text{rank}A} A, \text{ where } A \text{ is the} \\ \text{underlying vector bundle} \end{array} \right\}$

<u>Sketch of pf</u>

(i)

"Gerstenhaber $\Rightarrow$ Lie algebroid"

Assume $(\Gamma(\Lambda^\bullet A), \wedge, [\,,\,])$ is a Gerstenhaber alg.

<u>Bracket</u> = restriction of $[\,,\,]$ to $\Gamma(A)$

$$[\,,\,] : \Gamma(\Lambda^{\overset{1}{\overset{\shortparallel}{a}}} A) \times \Gamma(\Lambda^{\overset{1}{\overset{\shortparallel}{b}}} A) \to \Gamma(\Lambda^{\overset{1+1-1=1}{\overset{\shortparallel}{a+b-1}}} A)$$

$$\Gamma(A) \times \Gamma(A) \to \Gamma(A)$$

<u>Anchor</u>:

$$\rho : \Gamma(A) \to \Gamma(T_M)$$

$\rho_X(f) := [X, f]$

One checks that $(A, [\,,\,], \rho)$ is a Lie algebroid.

"Lie algebroid $\Rightarrow$ Gerstenhaber"

Just extend

"$D \Rightarrow$ cntn"

$D : \Gamma(\Lambda^\bullet A) \to \Gamma(\Lambda^{\bullet-1} A)$

(ii) Assume $(\Gamma(\Lambda^\bullet A), \wedge, [\,,\,], D)$ is a BV alg. Define

$r = \operatorname{rank} A$

$$\nabla : \underset{X}{\Gamma(A)} \times \underset{\Omega}{\Gamma(\Lambda^r A)} \to \Gamma(\Lambda^r A)$$

$$\nabla_X \Omega = - X \wedge \underbrace{D\Omega}_{\in \Gamma(\Lambda^{r-1} A)}$$

Claim

This defines a connection

pf

- $\nabla_{fX} \Omega = -(fX) \wedge D\Omega = f \cdot (-X \wedge D\Omega)$
$= f \nabla_X \Omega$

- $\nabla_X (f\Omega) = ?$

To show it, note that since $X\wedge\Omega=0$,

$$[X,\Omega] = (-1)^{|X|}\Big(\overset{=0}{\cancel{D(X\wedge\Omega)}} - (DX)\wedge\Omega - \overset{=-1}{\boxed{(-1)^{|X|}}}\,X\wedge D\Omega\Big)$$

$$\Rightarrow \nabla_X\Omega = -X\wedge D\Omega = \boxed{[X,\Omega]-(DX)\wedge\Omega}$$

So

$$\nabla_X(f\Omega) = [X,f\Omega] - (DX)\wedge(f\Omega)$$

$$= [X,f]\wedge\Omega + f\wedge[X,\Omega] - f\cdot(DX)\wedge\Omega$$

$$= \rho_X(f)\cdot\Omega + f\cdot\nabla_X\Omega \qquad \#$$

" Grth $\Rightarrow$ D "

Let D be an A-connection on $\Lambda^r A$

Define

$$D:\Gamma(\Lambda^k A)\to\Gamma(\Lambda^{k-1}A)$$

as follows.

Let $U\in\Gamma(\Lambda^k A)$ and write locally as

$$U=\iota_\omega\Omega$$

where $\omega\in\Gamma(\Lambda^{r-k}A^*)$, $\Omega\in\Gamma(\Lambda^r A)$.

$\Gamma(\Lambda^{k-1}A)$

For $m\in M$, set

$$(DU)|_m := -(-1)^{|\omega|}\Big(\underbrace{\iota_{d_A\omega}}_{\in\Gamma(\Lambda^{r-k+1}A)} \overbrace{\Omega}^{\in\Gamma(\Lambda^r A)} + \sum_{i=1}^{r} \iota_{\alpha_i\wedge\omega}(\nabla_{X_i}\Omega)\Big)$$

where $\{X_1, \cdots, X_r\}$ is a basis for $A|_m$, and $\{\alpha_1, \cdots, \alpha_r\}$ is its dual basis.

exer: D is well-defined.

"flat" Check: for $U \in \Gamma(\Lambda^\bullet A)$

$$D^2 U = -\iota_R U$$

Then check other conditions of BV algebras

$\cdots$ exercise

#

Remark For $X \in \Gamma(A)$, define the (generalized) divergence of X to be

$$\mathrm{div}_\nabla X := DX$$

exer:

If $A = T_M$, ω = a volume form on M, $\in\Gamma(\Lambda^{\dim M} T_M^*)$

Choose $\rightsquigarrow$ a flat conn D on ΛT_M s.t

$$\langle \nabla_X \omega, \Omega \rangle = X\langle \omega, \Omega \rangle - \langle \omega, \nabla_X \Omega \rangle = 0$$

$$\rightsquigarrow \quad \operatorname{div}_\nabla X = DX$$

Check this gives the classical divergence.

Atiyah classes of Lie pairs and dg Lie algebroids

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Introduction

$\Phi \to (\mathfrak{h}, \mu)$ $\quad \Phi_\omega \to \mathfrak{h}_\omega$

$\mathfrak{g}$: L_∞ alg λ: $\rightsquigarrow$ $\mathfrak{g}_\omega$, $\lambda_i^\omega(x_1, \dots, x_i) = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_{i+k}(x_1, \dots, x_k, \omega, \dots, \omega)$

$\omega \in MC(\mathfrak{g})$

- **Origin:** Atiyah (1957) constructed Atiyah class to characterize the obstruction to the existence of holomorphic connections on a holomorphic vector bundle

  v.s. iso.

  PBW: $S\mathfrak{g} \to \mathcal{U}\mathfrak{g}$ $\quad \mathfrak{g}$"

  PBW$\circ J^{\frac{1}{2}}$: $(S\mathfrak{g})^{\mathfrak{g}} \to (\mathcal{U}\mathfrak{g})^{\mathfrak{g}}$ $\quad Z(\mathcal{U}\mathfrak{g})$

- **Motivations of studying Atiyah classes:**
  - Kontsevich (1997): relation between Atiyah class in complex geometry and Duflo isomorphism in Lie theory

    alg iso.

    $\mathfrak{g}^*$ - Poisson mfld

  - Kapranov (1997): Atiyah class of a Kähler manifold induces an L_∞ structure — important in Rozansky–Witten invariants

    Let $L \to M$ be a Lie algebroid

    $A \to M$ be its subalgebroid

- **Beyond complex geometry:**
  - Chen–Stiénon–Xu (2012): extended the theory of Atiyah class from complex geometry to Lie (algebroid) pairs

    (L, A) is called a Lie pair

  - Mehta–Stiénon–Xu (2015): extended the theory of Atiyah class from complex geometry to dg Lie algebroids

Question:

What is the precise relation between Atiyah classes of Lie pairs and Atiyah classes of dg Lie algebroids?

(L,A): Lie pair $\xrightarrow[\text{construction}]{\text{some}}$ $\mathcal{F}$ Fedosov dg Lie algebroid
$\downarrow$
$\mathcal{M}$

Answers:

- L–Stiénon–Xu (~~2016~~ 2019), Batakidis–Voglaire (~~2016~~ 2018): $\mathrm{Atiyah}(\mathcal{F}) \cong \mathrm{Atiyah}(L,A)$
  The Atiyah class of the associated Fedosov dg Lie algebroid can be identified with the Atiyah class of a given Lie pair.
- L (~~2022~~ 2023):
  The Atiyah class of the associated pullback dg Lie algebroid can be identified with the Atiyah class of a given Lie pair.
  The second construction is much simpler than the first one.

Today: Explain the second answer.

Lie pair

- Lie algebroid = vector bundle $L \to M$ + Lie bracket on $\Gamma(L)$ + vector bundle map (called the anchor) $\rho : L \to T_M$ + a compatibility condition
- Lie pair (L, A) = Lie algebroid $L \to M$ + Lie subalgebroid $A \to M$ of L.

Example

- If $\mathfrak{h}$ is Lie subalgebra of a Lie algebra $\mathfrak{g}$, then $(\mathfrak{g}, \mathfrak{h})$ is a Lie pair over a point.
- If F is a regular foliation on a manifold M, then (T_M, T_F) is a Lie pair over M.
- If X is a complex manifold, then $(T_X \otimes \mathbb{C}, T_X^{0,1})$ is a Lie pair over X.
- If $\mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebra action on a manifold M, then $((\mathfrak{g} \ltimes M) \bowtie T_M, \mathfrak{g} \ltimes M)$ is a Lie pair over M.

Chevalley–Eilenberg differential and covariant derivative

- $L =$ Lie algebroid over M, $\quad E =$ a vector bundle over M
- Chevalley–Eilenberg differential $d_L : \Gamma(\Lambda^k L^\vee) \to \Gamma(\Lambda^{k+1} L^\vee)$,

$$(d_L\omega)(l_0, \cdots, l_k) = \sum_{i=0}^{k} (-1)^i \rho_{l_i}(\omega(l_0, \cdots \widehat{l_i} \cdots, l_k)) \qquad \Gamma(\Lambda^\bullet T_M^* \otimes E)$$

$e \in \Gamma(E) = \Omega^0(M,E)$

$$\sum_{i<j} (-1)^{i+j} \omega([l_i, l_j], l_0, \cdots \widehat{l_i} \cdots \widehat{l_j} \cdots, l_k)$$

$X \in \Gamma(T_M)$ $\quad \iota_X (d^\nabla e) = \nabla_X e \in \Gamma(E)$, $\quad d^\nabla e \in \Omega^1(M,E)$

Recall $d^\nabla : \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E)$, $(d^\nabla)^2 = R^\nabla \wedge -$

- $d_L \circ d_L = 0$. One gets the Lie algebroid cohomology.
  - If $L = \mathfrak{g}$, then one gets the Lie algebra cohomology.
  - If $L = T_M$, then one gets the de Rham cohomology.
  - If $L = T_X^{0,1}$, then one gets the Dolbeault cohomology.
- Let $\nabla : \Gamma(L) \times \Gamma(E) \to \Gamma(E)$ be an L-connection on E.
- Covariant derivative is $d_L^\nabla : \Gamma(\Lambda^p L^\vee \otimes E) \to \Gamma(\Lambda^{p+1} L^\vee \otimes E)$, defined by the derivation rule and $\iota_l(d_L^\nabla(e)) = \nabla_l e$.

Atiyah classes of Lie pairs

- $(L, A) =$ a Lie pair over M. $\mathfrak{q} : L \to B = L/A$. ← quotient vector bundle
- Bott connection is an A-connection on B defined by

$$\nabla_a^{\mathrm{Bott}}(\mathfrak{q}(l)) = \mathfrak{q}([a, l]), \qquad \forall a \in \Gamma(A), l \in \Gamma(L).$$

- Bott connection $\nabla = \nabla^{\mathrm{Bott}}$ is flat because

$$\begin{aligned} R^{\nabla}(a_1, a_2)(\mathfrak{q}(l)) &= \nabla_{a_1}\nabla_{a_2}(\mathfrak{q}(l)) - \nabla_{a_2}\nabla_{a_1}(\mathfrak{q}(l)) - \nabla_{[a_1,a_2]}(\mathfrak{q}(l)) \\ &= \mathfrak{q}([a_1, [a_2, l]] - [a_2, [a_1, l]] - [[a_1, a_2], l]) \\ &= 0. \qquad (\because \text{Jacobi identity}) \end{aligned}$$

- Bott connection induces an A-module structure on $B^\vee \otimes \operatorname{End} B$ so that we have a cochain complex

$$d_{\mathrm{CE}} = d^{\nabla^{\mathrm{Bott}}} : \Gamma(\Lambda^\bullet A^\vee \otimes \underbrace{B^\vee \otimes \operatorname{End} B}_{\mathcal{B}}) \to \Gamma(\Lambda^{\bullet+1} A^\vee \otimes \underbrace{B^\vee \otimes \operatorname{End} B}_{\mathcal{B}}).$$

$$\left(d^{\nabla^{\mathrm{Bott}}}\right)^2 = 0$$

An L-connection ∇ on B extending the Bott connection, i.e.

$$\nabla_{i(a)}(\mathfrak{q}(l)) = \nabla_a^{\mathrm{Bott}}(\mathfrak{q}(l)) = \mathfrak{q}([a, l]), \quad \forall a \in \Gamma(A), l \in \Gamma(L),$$

$R^\nabla: L \otimes L \to \mathrm{End}\, B$ $\quad i: A \to L$

induces a bundle map $R_{1,1}^\nabla : A \otimes B \to \mathrm{End}\, B$,

$$R_{1,1}^\nabla(a, \mathfrak{q}(l)) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{[a,l]} \in \mathrm{End}\, B$$

Proposition (Chen–Stiénon–Xu)

The section

$$R_{1,1}^\nabla \in \Gamma(A^\vee \otimes B^\vee \otimes \mathrm{End}\, B)$$

is a 1-cocycle whose induced cohomology class

$$\alpha_{L/A} \in H^1_{\mathrm{CE}}(A, B^\vee \otimes \mathrm{End}\, B)$$

is independent of the choice of L-connection ∇ extending the Bott connection.

- $R^{\nabla}_{1,1}$ = Atiyah cocycle associated with ∇.
- $\alpha_{L/A}$ = Atiyah class of the Lie pair (L, A).
- If $(L, A) = (T_X \otimes \mathbb{C}, T_X^{0,1})$, then one gets the Atiyah class of complex manifold X:

$$\alpha_X \in H^{1,1}(X, \mathrm{End}(T_X^{1,0})).$$

- If $(L, A) = (T_M, T_F)$ (F is a regular foliation of M), then one gets the Molino's class

$$\alpha_F \in H^1_{\mathrm{dR}}(F, T_F^{\perp} \otimes \mathrm{End}(N_F)).$$

($N_{\mathcal{F}} = T_M/T_F$ is the normal bundle)
Atiyah cocycle is in $\Gamma(T_F^{\vee} \otimes T_F^{\perp} \otimes \mathrm{End}(N_F))$.

Differential graded manifolds

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with support M is a sheaf $\mathcal{R}$ of $\mathbb{Z}$-graded commutative C^∞_M-algebras over a smooth manifold M that is locally isomorphic to sheafication of $C^\infty_U \otimes S(V^\vee)$ for sufficiently small open subsets U of M and a certain finite-dimensional $\mathbb{Z}$-graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V^i$. Here $S(V^\vee)$ denotes the graded algebra of polynomials on V.

$$C^\infty(\mathcal{M}) := \mathcal{R}(\mathcal{M})$$

Definition

A dg manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a vector field $Q \in \mathfrak{X}(\mathcal{M}) = \mathrm{Der}(C^\infty(\mathcal{M}))$ of degree $+1$ such that $[Q, Q] = 2\, Q \circ Q = 0$.

Example

- If $\mathfrak{g}$ is a Lie algebra, then $(\mathcal{M} = \mathfrak{g}[1], Q = d_{\mathrm{CE}})$ is a dg manifold: $(C^\infty(\mathcal{M}), Q)$ = Lie algebra cochain complex
- If M is a smooth manifold, then $(\mathcal{M} = T_M[1], Q = d_{\mathrm{dR}})$ is a dg manifold: $(C^\infty(\mathcal{M}), Q)$ = de Rham complex
- If X is a complex manifold, then $(\mathcal{M} = T_X^{0,1}[1], Q = \bar{\partial})$ is a dg manifold: $(C^\infty(\mathcal{M}), Q)$ = Dolbeault complex
- If $L \to M$ is a Lie algebroid, then $(\mathcal{M} = L[1], Q = d_L)$ is a dg manifold: $(C^\infty(\mathcal{M}), Q)$ = Lie algebroid cochain complex
  $$(C^\infty(L[1]), d_L) \;=\; \cdots \xrightarrow{d_L} \Gamma(\Lambda^k L^\vee) \xrightarrow{d_L} \Gamma(\Lambda^{k+1} L^\vee) \xrightarrow{d_L} \cdots$$
- If s be a section of a vector bundle $E \to M$, then $(E[-1], i_s)$ is a dg manifold: derived zero locus of s.
  $$(C^\infty(E[-1]), i_s) \quad = \quad \cdots \xrightarrow{i_s} \Lambda^2 E^\vee \xrightarrow{i_s} \Lambda^1 E^\vee \xrightarrow{i_s} C^\infty(M)$$

Pullback dg Lie algebroid

- dg Lie algebroid $\mathcal{L} \to \mathcal{M}$ = Lie algebroid object in the category of dg manifolds
- The section space $(\Gamma(\mathcal{L}), \mathcal{Q})$ is a dg module over the dg algebra $(C^\infty(\mathcal{M}), Q)$.

Main example: the pullback dg Lie algebroid $\pi^! L \to A[1]$

$$\begin{tikzcd} \pi^! L \arrow[r, dashed] \arrow[d] & L \arrow[d] \\ A[1] \arrow[r] & M \end{tikzcd}$$

- (L, A) = Lie pair
- Pullback the Lie algebroid $L \to M$ via $\pi : A[1] \to M$

$$\begin{tikzcd} \pi^! L = T_{A[1]} \times_{T_M} L \arrow[r, dashed] \arrow[d, dashed] & L \arrow[d, "\rho_L"] \\ T_{A[1]} \arrow[r, "\pi_*"'] & T_M \end{tikzcd}$$

Remark: the usual pullback vector bundle $\pi^* L = A[1] \times_M L$ is NOT a Lie algebroid.

- $\pi^! L = T_{A[1]} \times_{T_M} L \to T_{A[1]} \to A[1]$ vector bundle over $A[1]$
- A section of $\pi^! L \to A[1]$ is of the form

$$(X, v), \quad X \in \mathfrak{X}(A[1]), v \in \Gamma(\pi^* L) = \{v : A[1] \to L \mid \pi_L \circ v = \pi\}$$

$(\partial, l\circ\pi)$ =

$$\begin{tikzcd} \pi^* L \arrow[r, dashed] \arrow[d] & L \arrow[d] \\ A[1] \arrow[r, "\pi"] \arrow[u, bend left, "l\circ\pi"] \arrow[r, bend right, "\pi"'] & M \arrow[u, bend right, "l"'] \end{tikzcd}$$

with $\pi_* X = \rho_L \circ v : A[1] \to T_M$.
- $\Gamma(\pi^* L) \cong C^\infty(A[1]) \otimes_{C^\infty(M)} \Gamma(L)$ is generated by

$$l \circ \pi, \qquad l \in \Gamma(L),$$

as a $C^\infty(A[1])$-module.
- Anchor $\rho : \pi^! L \to T_{A[1]}$ is the projection map

$$\rho : \pi^! L = T_{A[1]} \times_{T_M} L \to T_{A[1]}, \quad \rho(X, v) = X.$$

- Lie bracket on $\Gamma(\pi^! L)$ is determined by the axioms and

$$[(X, l \circ \pi), (X', l' \circ \pi)] = ([X, X'], [l, l'] \circ \pi).$$

The inclusion map $\mathfrak{i} : A \hookrightarrow L$ induces a map

$$\tilde{\mathfrak{i}} : A[1] \to A \to L.$$

The pair

$$s_{\mathfrak{i}} := (d_A, \tilde{\mathfrak{i}}) \in \mathfrak{X}(A[1]) \times \Gamma(\pi^* L)$$

defines a section $s_{\mathfrak{i}} \in \Gamma(\pi^! L)$.

Proposition (Stiénon–Vitagliano–Xu)

The Lie algebroid $\pi^! L \to A[1]$ is a dg Lie algebroid over the dg manifold $(A[1], d_A)$ whose section space is equipped with the differential

$$\mathcal{Q} = [s_{\mathfrak{i}}, -] : \Gamma(\pi^! L) \to \Gamma(\pi^! L).$$

Atiyah class of dg Lie algebroid

- $(\mathcal{L}, \mathcal{Q})$ = dg Lie algebroid over $(\mathcal{M}, Q)$
- ∇ = $\mathcal{L}$-connection on $\mathcal{L}$, i.e. $\nabla : \Gamma(\mathcal{L}) \otimes_{\Bbbk} \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L})$

$$\nabla_{fX} Y = f \nabla_X Y$$
$$\nabla_X(fY) = X(f) \cdot Y + (-1)^{|X||f|} f \nabla_X Y$$

for $f \in C^\infty(\mathcal{M})$, $X, Y \in \Gamma(\mathcal{L})$. $L_Q(\nabla - \nabla')$

- Atiyah cocycle: $\mathrm{at}_{\mathcal{L}}^{\nabla} \in \Gamma(\mathcal{L}^\vee \otimes \operatorname{End} \mathcal{L})$ given by

$L_Q \circ L_Q = \frac{1}{2}[L_Q, L_Q] = \frac{1}{2} L_{[Q,Q]} = 0$

$$\mathrm{at}_{\mathcal{L}}^{\nabla'} \quad \mathrm{at}_{\mathcal{L}}^{\nabla}(X, Y) = L_{\mathcal{Q}}(\nabla)(X, Y)$$
$$= \mathcal{Q}(\nabla_X Y) - \nabla_{\mathcal{Q}(X)} Y - (-1)^{|X|} \nabla_X(\mathcal{Q}(Y))$$

- Atiyah class: $\alpha_{\mathcal{L}} = [\mathrm{at}_{\mathcal{L}}^{\nabla}] \in H^1(\Gamma(\mathcal{L}^\vee \otimes \operatorname{End} \mathcal{L}), L_{\mathcal{Q}})$

Proposition (Mehta–Stiénon–Xu)

- *The Atiyah class* $\alpha_{\mathcal{L}} \in H^1(\Gamma(T_{\mathcal{L}}^{\vee} \otimes \operatorname{End} T_{\mathcal{L}}), \mathcal{Q})$ *does not depend on the choice of* $\mathcal{L}$*-connection* ∇.
- $\alpha_{\mathcal{L}} = 0 \quad \Leftrightarrow \quad \exists$ $\mathcal{Q}$*-invariant connection on* $\mathcal{L}$

Advantage of dg Lie algebroid: The definition and meaning of Atiyah class are conceptually simple.

Disadvantage: The space

$$\Gamma(\mathcal{L}^\vee \otimes \operatorname{End} \mathcal{L})$$

is over complicated in many classical examples!!

For example, if $\mathcal{L}$ is the tangent bundle of $\mathcal{M} = T_X^{0,1}[1]$, then

$$\Gamma(T_{\mathcal{M}}^\vee \otimes \operatorname{End} T_{\mathcal{M}}) \cong \Gamma(T^\vee_{T_X^{0,1}[1]} \otimes T^\vee_{T_X^{0,1}[1]} \otimes T_{T_X^{0,1}[1]}).$$

Question: Why do we get the classical Atiyah class?

Main theorem

Theorem (L)

Given any Lie pair (L, A), there is an isomorphism

$$H^1\big(\Gamma\big((\pi^! L)^\vee \otimes \operatorname{End}(\pi^! L)\big), \mathcal{Q}\big) \xrightarrow{\cong} H^1_{\mathrm{CE}}(A, B^\vee \otimes \operatorname{End} B)$$

sending the Atiyah class of dg Lie algebroid $\pi^! L$ to the Atiyah class of Lie pair (L, A).

Corollary (Chen–Xiang–Xu)

Given any complex manifold X, there is an isomorphism

$$H^1(\Gamma((T_{T_X^{0,1}[1]})^\vee \otimes \mathrm{End}(T_{T_X^{0,1}[1]})), \mathcal{Q}) \xrightarrow{\cong} H^{1,1}(X, \mathrm{End}(T_X^{1,0}))$$

sending the Atiyah class of dg manifold $(T_X^{0,1}[1], \bar\partial)$ to the Atiyah class of complex manifold X.

Corollary (Chen–Xiang–Xu)

Given any regular foliation F of a manifold M, there is an isomorphism

$$H^1(\Gamma((T_{T_F[1]})^\vee \otimes \mathrm{End}(T_{T_F[1]})), \mathcal{Q}) \xrightarrow{\cong} H^1_{\mathrm{dR}}(F, T_F^\perp \otimes \mathrm{End}(N_F))$$

sending the Atiyah class of dg manifold $(T_F[1], d_{\mathrm{dR}})$ to the Atiyah class of Lie pair (T_M, T_F).

About the proof of main theorem

A contraction is the data

$$\begin{tikzcd} (V, d_V) \arrow[r, hook, shift left, "\tau"] & (W, d_W) \arrow[l, two heads, shift left, "\sigma"] \arrow[loop right, "h"] \end{tikzcd}$$

such that

- (V, d_V) and (W, d_W) are cochain complexes
- τ and σ are cochain maps
- $h : W \to W$ is a map of degree -1, called homotopy operator
- the following equations hold:

$$\sigma\tau = \mathrm{id}_V, \qquad id_W - \tau\sigma = hd_W + d_W h,$$
$$\sigma h = 0, \qquad h\tau = 0, \qquad hh = 0.$$

Tensor trick: One has an induced contraction

$$\begin{tikzcd} ((V^\vee)^{\otimes p} \otimes V^{\otimes q}, D_V) \arrow[r, hook, shift left, "I"] & ((W^\vee)^{\otimes p} \otimes W^{\otimes q}, D_W) \arrow[l, two heads, shift left, "\Pi"] \arrow[loop right, "H"] \end{tikzcd}$$

Proposition (Stiénon–Vitagliano–Xu, L)

Given a Lie pair (L, A) and a splitting

$$\begin{tikzcd} 0 \arrow[r] & \pi^*A \arrow[r, hook, "i"'] & \pi^*L \arrow[l, dashed, bend right, "\mathfrak{p}"'] \arrow[r, two heads, "q"'] & \pi^*B \arrow[l, dashed, bend right, "j"'] \arrow[r] & 0, \end{tikzcd}$$

there is a contraction:

$$(\Gamma(\Lambda^\bullet A^\vee \otimes B), d^{\mathrm{Bott}}) \underset{q}{\overset{\tau}{\rightleftarrows}} (\Gamma(\pi^! L), \mathcal{Q}) \circlearrowright \tilde{\mathfrak{p}}$$

where $\tilde{\mathfrak{p}}$ is essentially $\mathfrak{p}$ up to some identifications, and

$$\tau = j - \tilde{\mathfrak{p}}\mathcal{Q}j : \Gamma(\pi^*B) \cong \Gamma(\Lambda^\bullet A^\vee \otimes B) \longrightarrow \Gamma(\pi^! L) \cong \Gamma(\pi^*A[1] \oplus \pi^*L).$$

$\pi : A[1] \to M$, $B = L/A$

Remark

By choosing a connection of A, one can decompose

$$T_{A[1]} \cong A[1] \times_M (A[1] \oplus T_M) \; (\cong A[1] \times_M A[1] \times_M T_M)$$

so that

$$\pi^! L = T_{A[1]} \times_{T_M} L \cong A[1] \times_M (A[1] \oplus L) = \pi^* A[1] \oplus \pi^* L.$$

By the tensor trick, we have

Corollary

Let $\mathcal{L} = \pi^! L$. There is a contraction:

$$\left(\Gamma(\Lambda^\bullet A^\vee \otimes B^\vee \otimes \overset{B^\vee \otimes B}{\overset{\cong}{\underline{\operatorname{End} B}}}), d^{\mathrm{Bott}}\right) \underset{\Pi}{\overset{I}{\rightleftarrows}} \left(\Gamma(\mathcal{L}^\vee \otimes \operatorname{End}\mathcal{L}), \mathcal{Q}\right) \circlearrowright H$$

where Π is given by the formula $\Pi(\Theta) = \mathfrak{q} \circ \Theta \circ (\tau \otimes \tau)$.

The quasi-isomorphism Π induces the isomorphism in main theorem:

$$\Pi_* : H^1\left(\Gamma\left((\pi^! L)^\vee \otimes \operatorname{End}(\pi^! L)\right), \mathcal{Q}\right) \xrightarrow{\cong} H^1_{\mathrm{CE}}(A, B^\vee \otimes \operatorname{End} B)$$

I proved that this map sends the Atiyah class of $\pi^! L$ to the Atiyah class of (L, A).

Summary

The diagram

$$\begin{tikzcd}
\text{Lie (algebroid) pairs} \arrow[r, "\pi^! L", "\text{myself}"'] & \text{dg Lie algebroids} \\
\text{regular foliations} \arrow[u, "{(T_M, T_{\mathcal{F}})}"] \arrow[r, "{(T_{\mathcal{F}}[1], d_{\mathrm{dR}})}", "\text{Chen–Xiang–Xu}"'] & \text{dg manifolds} \arrow[u, "T_{\mathcal{M}}"'] \\
\text{complex manifolds} \arrow[u, "T_{\mathcal{F}} = T_X^{0,1}"] \arrow[r, "{(T_X^{0,1}[1], \bar{\partial})}", "\text{Chen–Xiang–Xu}"'] & \text{dg manifolds} \arrow[u, "="']
\end{tikzcd}$$

induces identifications of the Atiyah classes.

Thank you!

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