Geometric models for noncommutative algebras

Fall 2025, week 12

Lie algebroid

Def

A Lie algebroid is a vector bundle $A \to M$ equipped with

(i) a bundle map $\rho : A \to T_M$, called the anchor $\leftrightarrow$ $C^\infty(M)$-linear $\rho : \Gamma(A) \to \Gamma(T_M)$

(ii) a Lie bracket on $\Gamma(A)$

$$[\ ,\ ] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$$

s.t. $[X, -]$

$$[X, fY] = \rho_X(f) \cdot Y + f[X, Y]$$

$\forall\ X, Y \in \Gamma(A),\ f \in C^\infty(M)$

Prop

If $(A \to M, \rho)$ is a Lie algebroid, then

$$\rho : \Gamma(A) \to \Gamma(T_M) = \mathfrak{X}(M)$$

is a Lie alg morphism i.e. $[\rho_X, \rho_Y]_{\mathfrak{X}(M)} = \rho_{[X,Y]_A}$

pf

For $X, Y, Z \in \Gamma(A)$, $f \in C^\infty(M)$

$$[[X,Y], fZ] = \rho_{[X,Y]}(f)\cdot Z + \cancel{f[[X,Y],Z]}$$

$$\rho_Y(f) Z + f[Y,Z]$$

$$[[Y,fZ],X] = -[X,[Y,fZ]]$$

$$+f[[Y,Z],X]$$

$$= -\rho_X\rho_Y(f)Z - \rho_Y(f)[X,Z] - \rho_X(f)[Y,Z] \cancel{- f[X,[Y,Z]]}$$

$$\rho_X(f) Z + f[X,Z]$$

$$[[fZ,X],Y] = [Y,[X,fZ]]$$

$$[[Z,X],Y]$$

$$+) \quad = \rho_Y\rho_X(f) Z + \rho_X(f)[Y,Z] + \rho_Y(f)[X,Z] + \cancel{f[Y,[X,Z]]}$$

$$0 = (\rho_{[X,Y]} - [\rho_X, \rho_Y])(f)\cdot Z$$

$$\forall Z \in \Gamma(A)$$

exer

$$\Rightarrow \quad \rho_{[X,Y]} - [\rho_X, \rho_Y] = 0 \qquad \#$$

exer

Let E be a vector bundle over M. $f \in C^\infty(M)$

Show that

$$f \cdot s = 0 \quad \forall s \in \Gamma(E)$$

$$\Rightarrow f = 0$$

Example

① Any Lie alg is a Lie algebroid over a one-point mfd

② $(T_M \to M, \rho = \mathrm{id}_{T_M})$ is a Lie algebroid.

③ A regular distribution $D \subseteq T_M$, i.e. D is a vector subbundle of T_M s.t. $\Gamma(D)$ is closed under $[\cdot,\cdot]_{T_M}$.

$\Rightarrow (D \to M, \rho = \text{inclusion})$ is a Lie algebroid.

④ Let $\mathfrak{g}$ be a finite-dim Lie alg with an action on a mfd M, i.e. a Lie alg. mor

$G \curvearrowright M$

$\frac{d}{dt} \exp(t\xi) \cdot m$

$\mathfrak{g} \to \mathfrak{X}(M), \quad a \mapsto \hat{a}$

$\mathfrak{g} \xrightarrow{\text{at } t=0}$
$\hat{a} = \frac{d}{dt}x(t)\big|_{t=0}$

$\Rightarrow$ action Lie algebroid $(\mathfrak{g} \ltimes M \to M, \rho)$:

- the underlying v.b. is $M \times \mathfrak{g} \to M$
- anchor $\rho : \Gamma(M \times \mathfrak{g} \to M) \to \Gamma(T_M)$

$f \otimes a \longmapsto f \cdot \hat{a}$

$\Gamma(M\times\mathfrak{g}\to M) \cong C^\infty(M) \otimes \mathfrak{g} \cong \{M \to \mathfrak{g} \text{ smooth functions}\}$, $a \in \mathfrak{g}$

- bracket $[\,,\,] : C^\infty(M)\otimes\mathfrak{g} \times C^\infty(M)\otimes\mathfrak{g} \to C^\infty(M)\otimes\mathfrak{g}$

$$[f\otimes a, g\otimes b] = \rho_{fa}(g)\cdot b + g\,[fa, b]$$

$$= f\rho_a(g)\cdot b - g\,\underline{[b, fa]} = \rho_b(f)\,a + f\,[b,a]$$

$$= f\,\hat{a}(g)\otimes b - g\,\hat{b}(f)\otimes a + fg\otimes \underbrace{[a,b]}_{=[a,b]_{\mathfrak{g}}}$$

⑤ Given a Poisson (M, Π), let

$$A = T^*_M \to M$$

$$\rho = \Pi^\# : T^*_M \to T_M, \quad \omega \mapsto \omega\Pi$$

For $\xi, \eta \in \Gamma(A) = \Omega^1(M)$,

$$[\xi, \eta] = L_{\Pi^\#\xi}(\eta) - L_{\Pi^\#\eta}(\xi) - d\big(\Pi(\xi,\eta)\big)$$

$\in \Omega^1(M)$

Then $(A = T^*_M \to M, \rho)$ is a Lie algebroid called the cotangent Lie algebroid associated with the Poisson mfd (M, π).

<u>exer</u>

Check this defines a Lie algebroid.

<u>Distribution associated with a Lie algebroid</u>

Let $(A \to M, \rho)$ be a Lie algebroid.

Consider the distribution S of M:

$$S_m := \rho(A|_m)$$

for $m \in M$.

Since ρ preserves Lie brackets, S is involutive

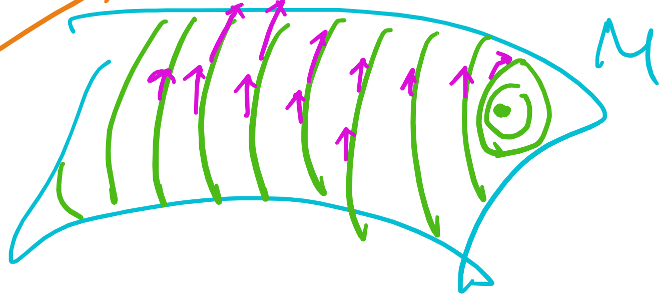
In fact, we have

Prop

If $(A \to M, \rho)$ is a Lie abd, then $\rho(A) \subseteq T_M$ defines a singular integrable distribution.

i.e. not locally constant rank

This distribution defines a singular foliation on M, whose leaves are called the orbits of A.



Recall that a leaf of the foliation induced by a distribution S is a maximal connected submfd of M whose tangent bundle is spanned by S

Example

① The orbits of a foliation Lie algebroid are the leaves of the regular foliation.

② The orbits of an action Lie algebroid are the usual orbits of the Lie alg action

③ The orbits of a cotanget Lie algebroid are the symplectic leaves of the Poisson mfd.

It is clear that

$$\rho(A|_m) = T_m M \big/ \ker(\rho|_m)$$

a Lie algebra, called the isotropy Lie algebra

<u>Prop</u>

If $(A \to M, \rho)$ is a Lie algebroid, then $\forall m \in M$, $\ker(\rho|_m) \subseteq A|_m$ carries a natural Lie alg structure.

pf

Let $m \in M$, $u, v \in \ker(\rho|_m)$

Let $X, Y \in \Gamma(A)$ s.t. $X|_m = u$, $Y|_m = v$

Define

$$[u,v] := [X,Y]|_m$$

<u>Claim</u>

$[u,v]$ does NOT depend on the choice of X and Y.

Let $Y_1 \in \Gamma(A)$ s.t. $Y_1|_m = v$.

$$\Rightarrow (Y - Y_1)_m = 0$$

<u>exer</u>

Let $E \to M$ be a vector bundle, $m \in M$.

If $s \in \Gamma(E)$, $s(m) = 0$, then $\exists f \in C^\infty(M)$ and $\exists s' \in \Gamma(E)$ s.t.

(i) $f(m) = 0$

<u>Recall</u>:

If $f \in C^\infty(\mathbb{R})$ with $f(0) = 0$,

then

(ii) $\delta = f \cdot \delta'$

then

$f(x) = x \cdot \int_0^1 f'(tx)\,dt$

$\Rightarrow \quad Y = Y_1 + f Y_2 \quad , \quad f \in C^\infty(M), \; f(m) = 0$

$\Rightarrow$

$$[X, Y]|_m = [X, Y_1]|_m + [X, fY_2]|_m$$

$$[X, fY_2]|_m = (\rho_X(f) \cdot Y_2)|_m + (f \cdot [X, Y_2])|_m$$

$$= \underbrace{\rho_X(f)(m)}_{=\rho_u(f) = 0} \cdot Y_2|_m + \underbrace{f(m)}_{=0} \cdot [X, Y_2]|_m$$

$X|_m = u \in \ker(\rho)$

$$= [X, Y_1]|_m$$

#

Lie algebroid cohomology

Let $(A \to M, \rho)$ be a Lie algebroid.

$$C^k(A) = C^k(A, \mathbb{R}) = \Gamma(\Lambda^k A^*)$$

and

$$d_A : C^k(A) \to C^{k+1}(A)$$

$\theta \in C^k(A) = \Gamma(\Lambda^k A^*)$

$X_0, \dots, X_k \in \Gamma(A)$

$$(d_A\Theta)(X_0, X_1, \cdots, X_k)$$

$$= \sum_{i=0}^{k} (-1)^i \rho_{X_i}\big(\Theta(X_0, \cdots \widehat{X_i}, \cdots X_k)\big)$$

$C^\infty(M)$

$$+ \sum_{i<j} (-1)^{i+j}\, \Theta([X_i, X_j], X_0, \cdots \widehat{X_i} \cdots \widehat{X_j} \cdots X_k)$$

$\Theta \in \Gamma(A^*)$ $\quad (d_A\Theta)(X,Y) = \rho_X(\Theta(Y)) - \rho_Y(\Theta(X)) - \Theta([X,Y])$

<u>exer</u>

$$d_A \circ d_A = 0$$

The cohomology of the cochain complex

$$(C^\bullet(A) = \Gamma(\Lambda^\bullet A^*),\ d_A)$$

is called the (Chevalley-Eilenberg) cohomology of the Lie algebroid A.

<u>Remark</u>

① If M = point, then one gets the classical Chevalley-Eilenberg cohomology of a Lie algebra.

② If $A = T_M$, then this is the deRham cohomology.

③ If $A = T_X^{0,1}$ for a complex mfd X, then this is the Dolbeault cohomology $H(\Omega_X^{0,\bullet}, \bar{\partial})$

Lie algebroid structures and cdga structures

Def

A commutative differential graded algebra (cdga) over a field $\mathbb{k}$ is graded $\mathbb{k}$-alg

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^n$$

equipped with a $\mathbb{k}$-linear map of deg $+1$

$$d : \mathcal{A}^n \to \mathcal{A}^{n+1},$$

called differential, satisfying

(i) $d \circ d = 0$

i.e. $a \in \mathcal{A}^{|a|}$ for some $|a| \in \mathbb{Z}$ degree of a

(ii) For homogeneous a, b

(ii) for homogeneous $a, b \in \mathcal{A}$,

$$d(ab) = d(a)\cdot b + (-1)^{|a|\cdot\overset{=1}{|d|}}\, a\cdot d(b)$$

(iii) Graded commutativity: $\forall$ homogeneous $a, b \in \mathcal{A}$,

$$a\cdot b = (-1)^{|a|\cdot|b|}\, b\cdot a$$

A morphism of cdga is a graded alg mor $\phi : (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ s.t.

$$\phi \circ d_{\mathcal{A}} = d_{\mathcal{B}} \circ \phi$$

Example (de Rham cx)

$$\mathcal{A}^n = \Omega^n(M)$$

$(\Omega^\bullet(M), d_{dR})$ is a cdga.

exer

Let $(A \to M, \rho)$ be a Lie algebroid,

$\Rightarrow$ $(C^\bullet(A), d_A)$ is a cdga.

$(\Gamma(\Lambda^\bullet A^*), d_A)$ is a cdga.

In fact, we also the converse:

<u>Prop</u>

Let $A \to M$ be a vector bundle. Then

$$\left\{ \begin{array}{l} \text{Lie algebroid} \\ \text{str on } A \end{array} \right\} \underset{\text{onto}}{\overset{1\text{-}1}{\longleftrightarrow}} \left\{ \begin{array}{l} \text{cdga str} \\ \text{on } \Gamma(\Lambda^\bullet A^*) \end{array} \right\}$$

$$(A, \rho, [\,,\,]) \longmapsto (\Gamma(\Lambda^\bullet A^*), d_A)$$

$$\begin{array}{l} \rho = ? \\ [\,,\,] = ? \end{array} \longleftarrow\!\!\mid \; (\Gamma(\Lambda^\bullet A^*), d)$$

<u>pf</u>

By the previous exercise,

$$(\rho, [\,,\,]) \longmapsto d_A, \quad (\Gamma(\Lambda^\bullet A^*), d_A) \text{ is a cdga.}$$

Conversely, given any

$$d : \Gamma(\Lambda^\bullet A^*) \to \Gamma(\Lambda^{\bullet+1} A^*)$$

s.t.

$$d(ab) = d(a)\cdot b + (-1)^{|a|} a \cdot d(b).$$

define

- $\rho : A \to TM$, $\quad X \in \Gamma(A)$, $f \in C^\infty(M)$

$$\rho_X(f) := \langle \underbrace{df}_{\in\, \Gamma(\Lambda^1 A^*) = \Gamma(A^*)}, \overbrace{X}^{\in\, \Gamma(A)} \rangle \in C^\infty(M)$$

- $[\,,\,] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is characterized by

$$\langle \theta, [X,Y] \rangle = \rho_X \langle \theta, Y \rangle - \rho_Y \langle \theta, X \rangle - (d\theta)(X,Y)$$

$\forall\, \theta \in \Gamma(A^*)$

It is clear that ρ is $C^\infty(M)$-linear

and $[X,Y] = -[Y,X]$

<u>Jacobi</u> :

<u>Claim 1</u> : For $\omega \in \Gamma(\Lambda^2 A^*)$,

$$(d\omega)(X,Y,Z) = \rho_X(\omega(Y,Z)) - \rho_Y(\omega(X,Z)) + \rho_Z(\omega(X,Y)) - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y)$$

<u>pf of Claim 1</u>

Since $\Gamma(\Lambda^2 A^*)$ is ... $\alpha \wedge \beta$ ... $\alpha, \beta \in \Gamma(A^*)$

since $\Gamma(\wedge^2 A^*)$ is spanned by $\alpha\wedge\beta$, $\alpha,\beta\in\Gamma(A^*)$,
it suffices to verify Claim1 for $\alpha\wedge\beta =\omega$.

Recall:
$(\xi\wedge\eta\wedge\zeta)(X,Y,Z)$
$= \xi(X)\eta(Y)\zeta(Z) - \xi(X)\eta(Z)\zeta(Y) + \cdots\cdots$

$(\Gamma(\wedge A^*), d)$ is a cdga

$$d(\alpha\wedge\beta)(X,Y,Z)$$
$$= (d\alpha\wedge\beta - \alpha\wedge d\beta)(X,Y,Z)$$
$$= (d\alpha)(X,Y)\cdot\beta(Z) + (d\alpha)(Y,Z)\beta(X) + (d\alpha)(Z,X)\beta(Y)$$
$$- \alpha(X)(d\beta)(Y,Z) - \alpha(Y)(d\beta)(Z,X) - \alpha(Z)(d\beta)(X,Y)$$

$$= \underbrace{(d\alpha)(X,Y)}_{\rho_X\alpha(Y) - \rho_Y\alpha(X) - \alpha([X,Y])}\cdot\beta(Z) - \alpha(X)\underbrace{(d\beta)(Y,Z)}_{\rho_Y\beta(Z) - \rho_Z\beta(Y) - \beta([Y,Z])}$$

$\rho_Y(\alpha(X)\cdot\beta(Z))$

$+$ cyclicly permuted terms

$$= \rho_Y(\alpha(Z)\beta(X) - \alpha(X)\beta(Z)) + \cdots\cdots$$
$$= \rho_X((\alpha\wedge\beta)(Y,Z)) - \rho_Y((\alpha\wedge\beta)(X,Z))$$
$$+ \rho_Z((\alpha\wedge\beta)(X,Y)) - (\alpha\wedge\beta)([X,Y],Z)$$
$$- (\alpha\wedge\beta)([Y,Z],X] - (\alpha\wedge\beta)([Z,X],Y)$$

$=$ RHS of Claim1 #

Claim2

$$\rho_{[X,Y]} = \rho_X\circ\rho_Y - \rho_Y\circ\rho_X$$

pf of Claim2

$$\rho_{[X,Y]}(f) = (df)([X,Y])$$

$$= \rho_X \underbrace{\langle df, Y\rangle}_{=\rho_Y(f)} - \rho_Y \underbrace{\langle df, X\rangle}_{\rho_X(f)} - (\underbrace{d(df)}_{=0 \text{ by cdga}})(X,Y)$$

$$= [\rho_X, \rho_Y](f) \qquad \dashv$$

pf of Jacobi $\qquad \Theta \in \Gamma(A^*)$

$$0 = (d(d\Theta))(X,Y,Z)$$

$$\overset{\text{claim1}}{=} \rho_X(d\Theta(Y,Z)) - \rho_Y(d\Theta(X,Z)) + \rho_Z(d\Theta(X,Y)) - d\Theta([X,Y],Z) - d\Theta([Y,Z],X) - d\Theta([Z,X],Y)$$

$$= \rho_X(\rho_Y \Theta(Z) - \rho_Z\Theta(Y) - \Theta([Y,Z])) - (\underbrace{\rho_{[X,Y]}}_{\rho_X\rho_Y - \rho_Y\rho_X}\Theta(Z) - \rho_Z\Theta([X,Y]) - \Theta([[X,Y],Z])) + \text{c.p.t.}$$

$$= \Theta([[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y])$$

$$= 0 \qquad \forall \Theta \in \Gamma(A^*)$$

$-0 \qquad\qquad \forall\, \theta \in \Gamma(A^*)$

$\Rightarrow \qquad \overset{\parallel}{0} \qquad \Rightarrow$ Jacobi #

It remains to prove:

$$[X, fY] = \rho_X(f)\cdot Y + f[X,Y]$$

Consider $\forall\, \theta \in \Gamma(A^*)$

$$\langle \theta, [X, fY] \rangle$$

$$= \rho_X\, \theta(fY) - \rho_{fY}\, \theta(X) - d\theta(X, fY)$$

$$\rho_X(f)\cdot \theta(Y) + f\cdot \rho_X(\theta(Y)) \qquad \langle \theta, [X,Y] \rangle$$

$$= \rho_X(f)\cdot\theta(Y) + f\cdot\big(\rho_X\, \theta(Y) - \rho_Y\, \theta(X) - d\theta(X,Y)\big)$$

$$= \langle \theta, f\cdot[X,Y] + \rho_X(f)\cdot Y \rangle$$

$\Rightarrow$ ok #