Geometric models for noncommutative algebras
Fall 2025, week 11

Recall

Let (M_1, Π_1) and (M_2, Π_2) be Poisson mfd

A (Poisson) relation $R : (M_1, \Pi_1) \to (M_2, \Pi_2)$ is a (coisotropic) submfd of $M_1 \times \overline{M_2}$.

If we have $R_1 : M_1 \to M_2$, $R_2 : M_2 \to M_3$,

then we have

$$R_2 \circ R_1 := \left\{ (x_1, x_3) \in M_1 \times M_3 \;\middle|\; \begin{array}{l} \exists\, x_2 \in M_2 \text{ s.t.} \\ (x_1, x_2) \in R_1, \\ (x_2, x_3) \in R_2 \end{array} \right\}$$

not necessarily a submfd of $M_1 \times M_3$

Def

Given $R_1 : M_1 \to M_2$, $R_2 : M_2 \to M_3$, their composition $R_2 \circ R_1$ is clean if

(i) $R_2 \circ R_1$ is a submfd of $M_1 \times M_3$

(ii) $T_{R_2 \circ R_1} = T_{R_2} \circ T_{R_1}$

Thm

If R_1, R_2 are Poisson relations s.t. $R_2 \circ R_1$ is clean, then $R_2 \circ R_1$ is also a Poisso relation

Since a Coisotropic submfd $C \subseteq M$, can be veiwed as a Poisson relation

$$\underset{\{x_0\}}{\overset{\shortparallel}{M_0}} \longrightarrow M_1 \qquad \left(\Leftrightarrow \text{coisotropic submfd of } M_0 \times \overline{M_1} = \overline{M_1} \right)$$

we have

Cor

If $R : M_1 \to M_2$ is a Poisson relation s.t. $R \circ C$ is clean, then

$$R \circ C = R(C) := \{ x_2 \in M_2 \mid \exists\, x_1 \in C \text{ s.t. } (x_1, x_2) \in R \} \subseteq M_2$$

is a Coisotropic submfd of M_2.

Deformation quatization

Recall that a Poisson mfd can be viewed as a math. model for classical mechanics.

Poisson mfd $\leftrightarrow$ classical mechanics

? $\leftrightarrow$ quantum mechanics

$\exists$ many approaches

Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer: proposed studying "quantum observables" by $(C^\infty(M)[[\hbar]], *)$

<u>Def</u> $(\mathbb{k} = \mathbb{R})$

Let M be a smooth mfd. A star product on M is a $\mathbb{k}[[\hbar]]$-bilinear associative multiplication

$= \{\sum_{k=0}^{\infty} f_k \cdot \hbar^k \mid f_k \in C^\infty(M)\}$

$* : C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$

$$\star : C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$$

that deforms the pointwise multiplication on $C^\infty(M)$

i.e. for $f, g \in C^\infty(M) \subseteq C^\infty(M)[[\hbar]]$

$$f \star g = \sum_{k=0}^{\infty} \underbrace{B_k(f,g)}_{C^\infty(M)} \cdot \hbar^k \in C^\infty(M)[[\hbar]]$$

satisfying

$$B_0(f,g) = f \cdot g$$

Furthermore, the coefficients B_k of $\star$ are required to satisfy the conditions:

- Each B_k is a bidifferential operator on M.

- $B_0(f,g) = f \cdot g$

  "$\lim_{\hbar \to 0} \dfrac{f \star g - g \star f}{\hbar}$"

  $B_1(f,g) - B_1(g,f) =: \{f,g\}$ is a Poisson bracket on M.

- $1 \star f = f \star 1 = f \quad \forall f \in C^\infty(M)$.

Deformation of algebras

Let A be an associative alg.

Def

An $\hbar$-linear product $*$ on $A[[\hbar]]$ is a bilinear operation

$$*: A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]]$$

s.t.

$$\left(\sum_{i=0}^{\infty} a_i \hbar^i\right) * \left(\sum_{j=0}^{\infty} b_j \hbar^j\right)$$

$$= \sum_{i,j=0}^{\infty} \left(a_i \hbar^i * b_j \hbar^j\right)$$

$$= \sum_{i,j=0}^{\infty} (a_i * b_j)\, \hbar^{i+j}$$

$$= \sum^{\infty} \left(\sum a_i * b_j \right) \hbar^n$$

$$= \sum_{n=0} \Big(\sum_{\substack{i+j=n \\ i,j \geq 0}} a_i * b_j \Big) \cdot \hbar^n$$

A deformation product of A is an associative $\hbar$-linear product $*$ on $A[[\hbar]]$ s.t.

$$(a*b)\Big|_{\hbar=0} = a \cdot b = m_A(a,b)$$

$$\| \qquad a \cdot b + \sum_{i=1}^{\infty} B_i(a,b)\, \hbar^n$$

$\forall\, a, b \in A$

Remark

An $\hbar$-linear product $*$ is uniquely determined by its retriction

$$*\Big|_{A \times A} : A \times A \longrightarrow A[[\hbar]]$$

$$a*b = \sum_{k=0}^{\infty} B_k(a,b) \cdot \hbar^k$$

$$= B_0(a,b) + B_1(a,b) \cdot \hbar + B_2(a,b)\, \hbar^2$$

$+ \cdots$

where $a, b \in A \subseteq A[[\hbar]]$

$B_k(a,b) \in A$

$B_k : A \times A \to A$ bilinear maps.

$*$ is a deformation product $\Leftrightarrow$ $B_0 = m_A$

Conversely, any seq

$B_k : A \times A \to A$

of bilinear maps, we can define

$$\left(\sum_{i=0}^{\infty} a_i \hbar^i\right) * \left(\sum_{j=0}^{\infty} b_j \hbar^j\right)$$

$$= \sum_{n=0}^{\infty} \Big(\sum_{i+j=n} \underbrace{(a_i * b_j)}_{= \sum_{k=0}^{\infty} B_k(a_i, b_j) \hbar^k} \Big) \hbar^n$$

$= \hbar^N$

$$= \sum_{N=0}^{\infty} \left(\sum_{i+j+k=N} B_k(a_i, b_j) \right) \cdot \hbar^N$$

$i, j, k \geq 0$ ⊛

However, this induced product $*$ is NOT necessarily associative.

Q:

$B_0 = m_A$

What seq. $(B_k)_{k=1}^{\infty}$ induces a deformation product?

i.e. ⊛ is associative or not?

That is, we shall check

① $$\left(\left(\sum_{i=0}^{\infty} a_i \hbar^i * \sum_{j=0}^{\infty} b_j \hbar^j\right) * \sum_{k=0}^{\infty} c_k \hbar^k\right)$$

and $\overset{?}{=}$

② $$\left(\sum_{i=0}^{\infty} a_i \hbar^i * \left(\sum_{j=0}^{\infty} b_j \hbar^j * \sum_{k=0}^{\infty} c_k \hbar^k\right)\right)$$

are equal, or not.

U

$$① = \sum_{N=0}^{\infty}\left(\sum_{\substack{i,j,p\geq 0\\ i+j+p=N}} B_p(a_i,b_j)\,\hbar^N\right) * \sum_{k=0}^{\infty} c_k \hbar^k$$

$$= \sum_{N=0}^{\infty}\sum_{i+j+p=N}\sum_{q=0}^{\infty} B_q(B_p(a_i,b_j),c_k)\cdot \hbar^{q+N+k}$$

$$= \sum_{N\geq 0} \boxed{\sum_{\substack{i,j,k,p,q\geq 0\\ i+j+k+p+q=N}} B_q(B_p(a_i,b_j),c_k)} \cdot \hbar^N$$

$B_0 = m_A$

$\|$

$(a_i b_j) c_k$

$$\sum_{\substack{i+j+k=N\\ p=q=0}} m_A(m_A(a_i,b_j),c_k)$$

$$+ \sum_{\substack{p\geq 1\\ i+j+k+p=N}} m_A(B_p(a_i,b_j),c_k)$$

$$+ \sum_{\substack{q\geq 1\\ i+j+k+q=N}} B_q(m_A(a_i,b_j),c_k)$$

$$+\sum_{\substack{p,q\geq 1\\ i+j+k+p+q=N}} B_q(B_p(a_i,b_j),c_k)$$

Similarly,

$$② = \sum_{N=0}^{\infty}\left(\sum_{\substack{i,j,k,p,q\geq 0\\ i+j+k+p+q=N}} B_q(a_i,B_p(b_j,c_k))\right)h^N$$

$\parallel \quad B_0=m_A$

$$\underbrace{\sum_{i+j+k=N} m_A(a_i,m_A(b_j,c_k))}_{a_i(b_jc_k)}+\sum_{\substack{p\geq 1\\ i+j+k+p=N}} m_A(a_i,B_p(b_j,c_k))$$

$$+\sum_{\substack{q\geq 1\\ i+j+k+q=N}} B_q(a_i,m_A(b_j,c_k))+\sum_{\substack{p,q\geq 1\\ i+j+k+p+q=N}} B_q(a_i,B_p(b_j,c_k))$$

So $*$ is associative $\Leftrightarrow$

$$0=\sum_{\substack{i,j,k,p,q\geq 0\\ i+j+k+p+q=N}} B_q(B_p(a_i,b_j),c_k)-B_q(a_i,B_p(b_j,c_k))$$

$B_0 = m_A$

$$= \sum_{\substack{i,j,k\geq 0,\, p\geq 1 \\ i+j+k+p=N}} m_A(B_p(a_i, b_j), c_k) - m_A(a_i, B_p(b_j, c_k))$$

$$+ \sum_{\substack{q\geq 1 \\ i+j+k+q=N}} B_q(m_A(a_i, b_j), c_k) - B_q(a_i, m_A(b_j, c_k))$$

$$+ \sum_{\substack{p,q\geq 1 \\ i+j+k+p+q=N}} B_q(B_p(a_i, b_j), c_k) - B_q(a_i, B_p(b_j, c_k))$$

$$= 0$$

<u>Example</u>

① $B_0 = m_A$, $B_1 = B_2 = \cdots = 0$

$\Rightarrow$ $*$ = natural multiplicative on $A[[h]]$

$$\sum a_i h^i \cdot \sum b_j h^j = \sum a_i b_j h^{i+j}$$

(associative)

② $B_0 = B_1 = m_A$, $B_2 = B_3 = \cdots = 0$

$$\Rightarrow \left(\sum_{i=0}^{\infty} a_i h^i\right) * \left(\sum_{j=0}^{\infty} b_j h^j\right)$$

$$= \sum_{N=0}^{\infty} \left(\sum_{i+j=N} a_i b_j + \sum_{i+j=N+1} a_i b_j \right) \cdot \hbar^N$$

which is also associative.

③ $B_0 = m_A$, $\quad B_1(a,b) = m_A(b,a) = ba$

$$\Rightarrow \left(\sum_{i=0}^{\infty} a_i \hbar^i\right) * \left(\sum_{j=0}^{\infty} b_j \hbar^j\right)$$

$$= \sum_{N=0}^{\infty} \left(\sum_{i+j=N} a_i b_j + \sum_{i+j=N-1} b_j a_i \right) \hbar^N$$

which is NOT associative (provided

A is noncommutative).

Hochschild cochains of algebras

Def

A Hochschild n-cochain of A is

an (n+1)-linear map

$$\Phi \in \mathrm{Hom}_{\Bbbk}(A^{\otimes n+1}, A)$$

$= C^n_H(A)$

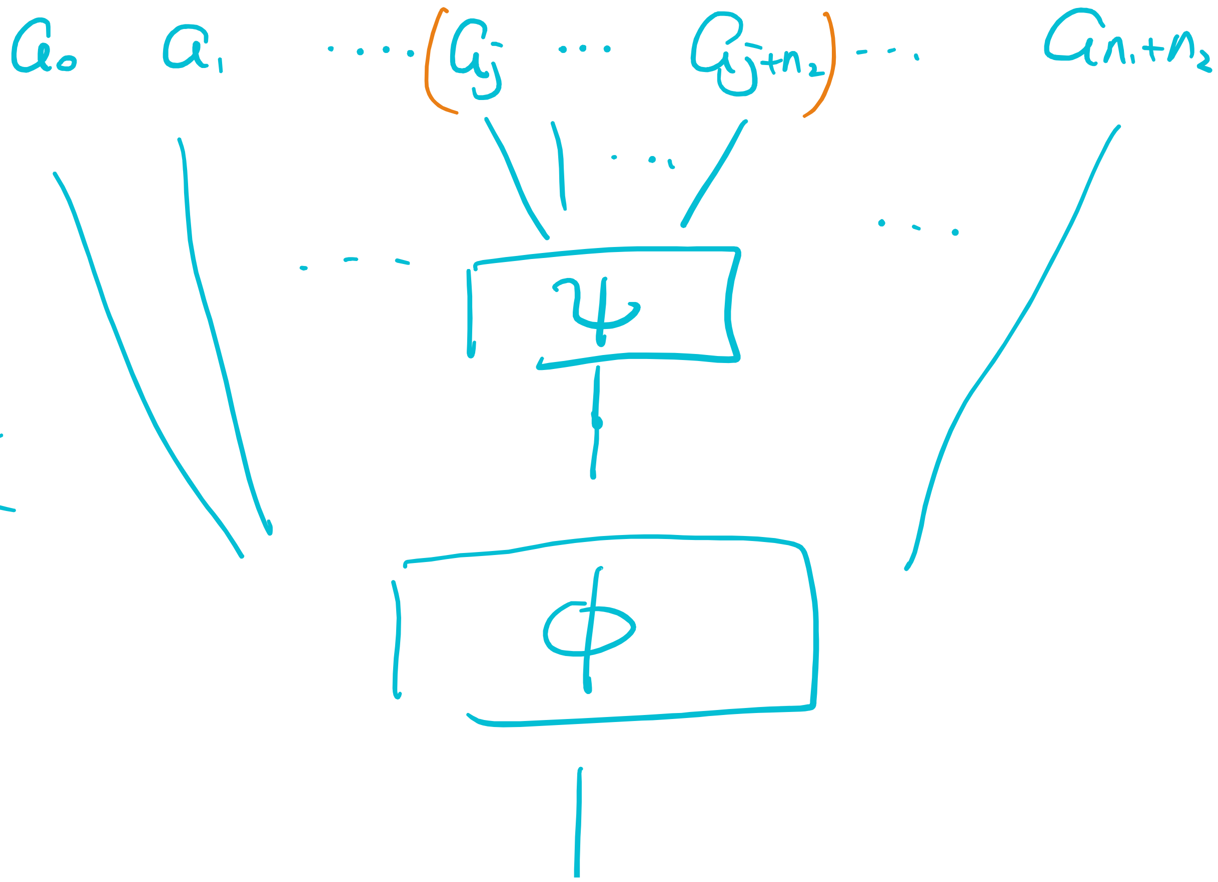
Define for $\phi \in C^{n_1}_H(A)$, $\psi \in C^{n_2}_H(A)$,

$\phi \bullet \psi \in C^{n_1+n_2}_H(A)$ by

$$(\phi \bullet \psi)(a_0, a_1, \ldots, a_{n_1+n_2})$$
$$:= \sum_{j=0}^{n_1} (-1)^{jn_2} \phi(a_0, \ldots, a_{j-1}, \psi(a_j, \ldots, a_{j+n_2}), a_{j+n_2+1}, \ldots, a_{n_1+n_2})$$

$\sum \pm$

all possible insertions



The Gerstenhaber bracket is

$$[\![\phi, \psi]\!] := \phi \bullet \psi - (-1)^{n_1 n_2} \psi \bullet \phi \in C_H^{n_1+n_2}(A)$$

Let $m_A \in C_H^1(A)$ be the multiplication of A.
The Hochschild differential is

$$d_H := [\![m_A, -]\!] : C_H^\bullet(A) \to C_H^{\bullet+1}(A)$$

Prop

$$(C_H^\bullet(A), d_H, [\![-,-]\!])$$

forms a dgla = differential graded Lie algebra

i.e.

(i) $[\![\ ,\ ]\!] : C_H^{n}(A) \times C_H^{m}(A) \to C_H^{n+m}(A)$

is a graded Lie bracket:

(i-1) $[\![\ ,\ ]\!]$ is bilinear

(i-2) (graded) skew-symmetry:

$$[\![\phi, \psi]\!] = -(-1)^{|\phi|\cdot|\psi|} [\![\psi, \phi]\!]$$

$\phi \in C_H^{|\phi|}(A)$, $\psi \in C_H^{|\psi|}(A)$

(i-3) (graded) Jacobi identity:

$$[\![\phi, [\![\psi, \sigma]\!]]\!]$$
$$= [\![[\![\phi, \psi]\!], \sigma]\!] + (-1)^{|\phi|\cdot|\psi|} [\![\psi, [\![\phi, \sigma]\!]]\!]$$

(ii) d_H is a differential compatible with $[\![\ ,\ ]\!]$

i.e.

(ii-1) Coboundary map:

$$d_H : C_H^{\bullet}(A) \to C_H^{\bullet+1}(A)$$

and

$$d_H \circ d_H = 0$$

(ii-2) Compatibility

$$d_H(\llbracket \phi, \psi \rrbracket) = \llbracket d_H(\phi), \psi \rrbracket + (-1)^{|\phi|} \llbracket \phi, d_H(\psi) \rrbracket$$

<u>Remark</u>

Let $B \in C^1_H(A) = \mathrm{Hom}_{\mathbb{K}}(A \otimes A, A)$ be a binary operation. Then

$$\begin{aligned}
\llbracket B, B \rrbracket (a_0, a_1, a_2) &= \left(B \bullet B - (-1)^{1 \cdot 1} B \bullet B\right)(a_0, a_1, a_2) \\
&= 2 (B \bullet B)(a_0, a_1, a_2) \\
&= 2\Big(B\big(B(a_0, a_1), a_2\big) - B\big(a_0, B(a_1, a_2)\big)\Big),
\end{aligned}$$

So

$$\llbracket B, B \rrbracket = 0 \iff B \text{ is associative}$$

In particular,

$$d_H^2(\phi) = \llbracket m_A, \llbracket m_A, \phi \rrbracket \rrbracket$$

$$= \frac{1}{2}[\![\cancel{[\![m_A, m_A]\!]}, \Phi]\!] + (-1)\underbrace{[\![m_A, [\![m_A, \Phi]\!]]\!]}_{-d_H^2(\Phi)}$$

$$\Rightarrow \quad d_H^2 = 0$$

Consider

$$C_H^\bullet(A)[\![\hbar]\!]$$

with

$$d_H\Big(\sum_{i=0}^{\infty} \phi_i \cdot \hbar^i\Big) = \sum_{i=0}^{\infty} d_H(\phi_i)\cdot \hbar^i$$

$$[\![\sum_{i=0}^{\infty}\phi_i\,\hbar^i, \sum_{j=0}^{\infty}\psi_j\,\hbar^j]\!]$$

$$= \sum_{N=0}^{\infty}\sum_{i+j=N}[\![\phi_i, \psi_j]\!]\cdot \hbar^N$$

$\Rightarrow$ we have a dgla

$$\big(C_H^\bullet(A)[\![\hbar]\!],\ d_H,\ [\![\ ,\ ]\!]\big)$$

Recall that a deformation product

$$*: A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!]$$

is defined

is determined by its restriction

$$*|_{A\times A} : A\times A \longrightarrow A[[\hbar]]$$

$$\|$$

$$\sum_{k=0}^{\infty} B_k\cdot\hbar^k = \underbrace{m_A + B_1\hbar + B_2\hbar^2 + \cdots}_{\in\, C^1_H(A)[[\hbar]]}$$

with $B_0 = m_A$

where $B_k \in C^1_H(A)$

The associativity of $*$ can be rephrased as follows:

$$\sum_{\substack{p,q\geq 0\\ p+q=N}} \overbrace{B_q(B_p(a,b),c) - B_q(a,B_p(b,c))}^{=\,(B_q\bullet B_p)(a,b,c)} = 0 \qquad \forall\, a,b,c\in A,\ N\geq 0$$

$$\Longleftrightarrow \quad \sum_{p+q=N} B_q\bullet B_p = 0 \qquad \forall\, N\geq 0$$

$$\Longleftrightarrow \quad \left(\sum_{q=0}^{\infty} B_q\cdot\hbar^q\right)\bullet\left(\sum_{p=0}^{\infty} B_p\cdot\hbar^p\right) = 0$$

$$\left(\sum_{N=0}^{\infty} \left(\sum_{p+q=N} B_q \bullet B_p \right) \hbar^N \right.$$

$$\frac{1}{2} [\![\underbrace{\sum_{p=0}^{\infty} B_p \hbar^p}_{\displaystyle = m_A + \underbrace{\sum_{p=1}^{\infty} B_p \hbar^p}_{=: \Theta}}, \sum_{p=0}^{\infty} B_p \hbar^p]\!] = 0$$

$$d_H(\Theta) = -(-1)^{1 \cdot 1} [\![m_A, \Theta]\!] = \underbrace{[\![\Theta, m_A]\!]}$$

$$\| $$

$$\frac{1}{2} \Big(\underbrace{[\![m_A, m_A]\!]}_{=0} + [\![m_A, \Theta]\!] + \underbrace{[\![\Theta, m_A]\!]} + [\![\Theta, \Theta]\!] \Big)$$

$$= \boxed{d_H(\Theta) + \frac{1}{2} [\![\Theta, \Theta]\!] = 0}$$

Maurer-Cartan equation (in a dgla)

Thm

A seq. $B_k \in C_H(A)$, $k=1,2,\cdots$,

induces a deformation product

$$\Leftrightarrow \quad \Theta = \sum_{k=1}^{\infty} B_k \hbar^k \in C_H^1(A)[[\hbar]]$$

is a Maurer-Cartan element (MC)

of the dgla

$$\left(C_H^\bullet(A)[[\hbar]],\ d_H,\ [[\ ,\ ]] \right)$$

That is,

$$d_H(\Theta) + \frac{1}{2}[[\Theta, \Theta]] = 0$$

Kontsevich's approach to star products

Main question:

Given a Poisson mfd $(M, \{,\})$,

$\exists ?$ a star product

$$*: C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \to C^\infty(M)[[\hbar]]$$

s.t. $\displaystyle\underbrace{\lim_{\hbar\to 0} \frac{1}{\hbar}(f*g - g*f)}_{B_1(f,g)-B_1(g,f)} = \{f,g\}$

$\forall\ f,g \in C^\infty(M)$

Kontsevich's idea:

star product $\Leftrightarrow$ MC elements in $(D_{poly}(M)[[\hbar]],\ d_H,\ [\ ,\ ])$

$\updownarrow$ Loo quasi-ISO

Poisson str $\to$ MC elements in $(T_{poly}(M)^{[[\hbar]]},\ 0,\ [\ ,\ ]_S)$

<u>Def</u>

The dgla $(T^i_{poly}(M), 0, [\ ,\ ])$ of polyvector fields on M is

$$T^i_{poly}(M) = \Gamma(\wedge^{i+1} T_M)\ ,\quad i=-1,0,1\ldots$$

$[\,,\,]$ = Schouten bracket

Remark

$\pi \in \Gamma(\wedge^2 T_M) = T^1_{poly}(M)$ is a Poisson str.

$\iff [\pi, \pi] = 0$

$\iff 0(\pi) + \frac{1}{2}[\pi, \pi] = 0$

$\iff \pi$ is a MC element.

Def

The dgla $(D^{\bullet}_{poly}(M), d_H, [\![\,,\,]\!])$ of polydifferential operators on M is a subdgla of $(C^{\bullet}_H(A), d_H, [\![\,,\,]\!])$

where

$$D^{\circ}_{poly}(M) = D(M) = \left\{ \begin{array}{l} \text{differential operators} \\ \text{on } M \end{array} \right\}$$

$C^\infty(M) \to C^\infty(M)$

// locally

$$\sum_I a_I(x) \frac{\partial^{i_1 + i_2 + \cdots + i_n}}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x^{i_n}}$$

$(i_1, i_2, \dots, i_n)$

$$D^{-1}_{poly}(M) = C^\infty(M) = R$$

$$D^{i}_{poly}(M) = \overbrace{D(M) \otimes_R D(M) \otimes_R \cdots \otimes_R D(M)}^{i+1}$$

$$\phi_0 \otimes \phi_1 \otimes \cdots \phi_i \in C^i_H(C^\infty(M))$$

$$f_0 \otimes \cdots \otimes f_i \mapsto \phi_0(f_0)\cdot \phi_1(f_1) \cdots \phi_i(f_i)$$

<u>Def</u> (HKR)

The Hochschild-Kostant-Rosenberg map

$$hkr : T^\bullet_{poly}(M) \longrightarrow D^\bullet_{poly}(M)$$

is

$$hkr(X_1 \wedge \cdots \wedge X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(k)}$$

$$(-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{'' odd} \end{cases}$$

S_k: permutation group of order k

$X_i \in \mathfrak{X}(M)$.

Thm (Kontsevich's formality thm).

$\exists$ L_∞ quasi-iso

$$\Phi : T^\bullet_{poly}(M) \rightsquigarrow D^\bullet_{poly}(M)$$

(i.e.,

$$\Phi_k : \Lambda^k T^\bullet_{poly}(M) \to D^\bullet_{poly}(M), \ k=1,2,\dots$$

satisfying certain conditions...)

s.t.

$$\Phi_1 = hkr : T^\bullet_{poly}(M) \to D^\bullet_{poly}(M).$$

<u>Remark</u>

An L_∞ morphism $\Phi : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ induces a map between MC elements:

$$\begin{array}{ccc} MC(\mathfrak{g}) & \longrightarrow & MC(\mathfrak{h}) \\ \Phi \\ \omega & \longmapsto & \displaystyle\sum_{k=1}^{\infty} \frac{1}{k!} \Phi_k(\underbrace{\omega \wedge \omega \wedge \dots \wedge \omega}_{k \text{ times}}) \end{array}$$

provided it converges

converges.

<u>Application to star products</u>:

Given a Poisson str Π on M,
Consider the MC element

$$\Pi \cdot \hbar \in MC\left(T_{poly}^{\bullet}(M)[[\hbar]], 0, [\,,\,]\right)$$

$$\big\downarrow \qquad\qquad\qquad\qquad \big\downarrow \ \Phi \text{ Kontsevich}$$

$$H \in MC\left(D_{poly}^{\bullet}(M)[[\hbar]], d_H, [\,,\,]\right)$$

$$\text{Thm} \ \big\updownarrow$$

$$* : C^{\infty}(M)[[\hbar]] \times C^{\infty}(M)[[\hbar]] \to C^{\infty}(M)[[\hbar]]$$

Furthermore, $H \in MC\left(D_{poly}^{\bullet}(M)[[\hbar]]\right)$

$\Rightarrow$ B_k are bidifferential operators

$\Phi_1 = hkr$

$$\Rightarrow \lim_{\hbar \to 0} \frac{1}{\hbar}\left(f * g - g * f\right) = \{f, g\}$$

So we can obtain a star product by Kontsevich's quasi-iso $\underline{\overline{\Phi}}$.