Geometric models for noncommutative algebras
Fall 2025, week 10

Let (M, Π) be a Poisson mfd, $x_0 \in M$.

Assume $\Pi|_{x_0} = 0$

Choose a local chart $(y_1, \cdots, y_n)$ with $\overset{x_0}{\underset{(0,\cdots,0)}{\updownarrow}}$

$$\phi_{ij} := \{y_i, y_j\} = \Pi_{ij}$$

$$\Rightarrow \quad \phi_{ij}(0, \cdots, 0) = 0 \qquad \forall\, i, j.$$

Expand ϕ_{ij}:

$$\phi_{ij}(y) = \overset{0}{\overset{\|}{\phi_{ij}(0)}} + \underbrace{\sum_{k=1}^{n} \overset{C_{ij}^{k}}{\overset{\|}{\boxed{\left.\frac{\partial \phi_{ij}}{\partial y_k}\right|_0}}} \cdot y_k + O(y^2)}$$

<u>Lemma</u>

$$C_{ij}^{k} = -C_{ji}^{k}$$

$$\sum_{l=1}^{n} \left(C_{ij}^{l} C_{lk}^{m} + C_{jk}^{l} C_{li}^{m} + C_{ki}^{l} C_{lj}^{m} \right) = 0$$

$\Rightarrow$ C_{ij}^k are the structure constants of a Lie alg $\mathfrak{g}$.

So if $\Pi|_{x_0} = 0$, then we have an associated Lie alg. $\mathfrak{g}$

<u>Recall</u>:

If $\mathfrak{g}$ is a finite-dim. Lie alg, then $\mathfrak{g}^*$ is a Poisson mfd. with

$$\{y_i, y_j\} = \sum_{k=1}^{n} C_{ij}^k y_k$$

where $\{y_1, \dots, y_n\}$ is a basis for $\mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$

<u>Def</u>

$\underset{}{(\mathfrak{g}^*, \{\ ,\ \})} = (\mathfrak{g}^*, \Pi_{\mathfrak{g}^*})$ is the linearization of (M, Π) at x_0

(M, Π) is <u>linearizable</u> at x_0 if it is equivalent to its linearization.

i.e. $\exists$ nbd U of x_0 $\exists$ Poisson diffeomorphism

$$(U, \Pi|_U) \underset{\text{Poisson map}}{\cong} (\mathfrak{g}^*, \Pi_{\mathfrak{g}^*})$$

<u>Example</u>

$\mathbb{R}^2$ with Poisson str

$$\Pi = \begin{pmatrix} 0 & y_1^2 + y_2^2 \\ -(y_1^2 + y_2^2) & 0 \end{pmatrix}$$

$\Rightarrow \Pi|_{(0,0)} = 0$ ~~HS~~

$\Rightarrow$ linearization $= (\mathbb{R}^2, 0)$

<u>Thm</u> (Conn) $x_0 \in M$

Let (M, Π) be a Poisson mfd $\Pi|_{x_0} = 0$

$$C_{ij}^k = \frac{\partial \{y_i, y_j\}}{\partial y_k}\Big|_0 \quad \text{as before}$$

If (C_{ij}^k) are the structure constants of a compact semisimple Lie alg. then π is linearizable

Symplectic foliation

Let (M, π) be a Poisson mfd.
Define the characteristic distribution S by

$$S = \pi^\#(T_M^*) \subseteq T_M$$

Recall: $\pi \in \Gamma(\Lambda^2 T_M)$,

$$\rightsquigarrow \pi^\# : T_M^* \longrightarrow T_M, \quad \pi^\#(\alpha) = \iota_\alpha \pi$$

Prop

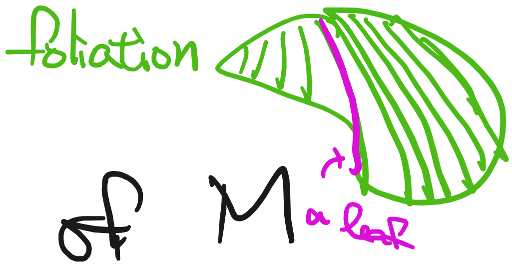
closed under $[\ ,\ ]$

S is smooth and involutive, and its

rank at $x \in M \overset{\text{def}}{=} \dim(S_x) \underset{\text{Prop}}{=} \operatorname{rank}(\Pi|_x)$

Furthermore, S is (completely) integrable.

This proposition will be studied in the framework of Lie algebroids



$\Rightarrow$ We get a foliation of M

a partition of M (by submfd)

Here a leaf of this foliation is a

L

maximal submfd L of M s.t.

$$T_x L = \Pi^{\#}(T_x^* M) \quad \forall x \in L.$$

This foliation is called the symplectic foliation of (M, Π) due to the following thm:

Thm

The given Poisson str Π induces a symplectic str. on each leaf of the foliation associated with the characteristic distribution.

$\Rightarrow$ Intuitively, any Poisson mfd can be decomposed into a disjoint union of symplectic mfds.

pf

Let L be a leaf of this foliation

$\Rightarrow$

$$T_x L = S_x = \Pi^\#(T_x^* M) \quad \forall x \in L$$

$$(\Rightarrow \quad \Pi^\#(T^*_M)|_L \subseteq T_L)$$

$\Rightarrow$ L is a Poisson submfd of M

" Π "

''L

Recall:

L is symplectic

$\Leftrightarrow \quad (\pi_L)^\# : T_x^* L \to T_x L$

is an iso. $\forall x \in L$

Since $\pi_L^\# (T_x^* L) = \pi^\# (T_x^* M)$

$= T_x L$

and $\dim T_x^* L = \dim T_x L$

we conclude $\pi_L^\#$ are iso

$\Rightarrow L$ is symplectic. #

<u>Application to $\mathfrak{g}^*$:</u>

Let $\mathfrak{g}$ be a Lie algebra.

$\forall X \in \mathfrak{g}$, we have $\phi_X \in C^\infty(\mathfrak{g}^*)$

$\forall X \in \mathfrak{g}$, we have $l_X \in C^\infty(\mathfrak{g}^*)$,

$$l_X(\xi) = \xi(X) \qquad \forall \xi \in \mathfrak{g}^*$$

define $\qquad \{l_X\} \subsetneqq C^\infty(\mathfrak{g}^*)$

$$\{l_X, l_Y\} := l_{[X,Y]_{\mathfrak{g}}} \qquad \xi \in \mathfrak{g}^*$$

This extends to $C^\infty(\mathfrak{g}^*)$ as follows:

The <u>Lie-Poisson bracket</u> on $C^\infty(\mathfrak{g}^*) \ni f, g$ is

$$\{f, g\}(\xi) = \xi\left([df|_\xi, dg|_\xi]_{\mathfrak{g}}\right)$$

where $\qquad$ de Rham

$$\underset{=}{d}f|_\xi, \ \underset{=}{d}g|_\xi \in T^*_\xi(\mathfrak{g}^*) \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$$

$\Rightarrow$ We have a Poisson mfd $\mathfrak{g}^*$

<u>Thm</u>

The symplectic leaves of $\mathfrak{g}^*$ are

The symplectic leaves of $\mathfrak{g}^*$ are

exactly the coadjoint orbits.

pf

If L is a symplectic leaf, then $x \in L$

$$T_xL = \pi^\#(T_x^*M) = \left\{ \underbrace{\pi^\#(df|_x)}_{= X_f|_x} \;\middle|\; f \in C^\infty(M) \right\}$$

Let $X, Y \in \mathfrak{g}$. We have $x = \xi \in \mathfrak{g}^* = M$

$$\ell_X, \ell_Y \in C^\infty(\mathfrak{g}^*)$$

One can check

$$Y = d(\ell_Y)|_\xi \in \mathfrak{g} \cong (\mathfrak{g}^*)^* \cong T_\xi^*(\mathfrak{g}^*)$$

$\in T_\xi(\mathfrak{g}^*) \cong \mathfrak{g}^*$

$$\Rightarrow \langle X_{\ell_X}|_\xi, Y \rangle = \langle X_{\ell_X}|_\xi, d(\ell_Y)|_\xi \rangle$$

$$= d(\ell_Y)(X_{\ell_X})|_\xi$$

$(df)(X) = X(f)$

$$= X_{\ell_X}(\ell_Y)|_\xi = \{\ell_X, \ell_Y\}|$$

$$= X_{\ell_X}(\ell_Y)|_\xi = \{\ell_X, \ell_Y\}|_\xi$$

$$= \ell_{[X,Y]_{\mathfrak{g}}}(\xi) = \xi(\underbrace{[X,Y]_{\mathfrak{g}}}_{= ad_X(Y)})$$

$$= -ad_X^*(\xi)(Y).$$

So

$$X_{\ell_X}|_\xi = -ad_X^*(\xi)$$

Let $S = \pi^\#(T\mathfrak{g}^*)$

$$\Rightarrow \quad S_\xi = \{\pi^\#(df)|_\xi \mid f \in C^\infty(\mathfrak{g}^*)\}$$

$$= \{\overbrace{\underbrace{\pi^\#(d\ell_X)}_{= X_{\ell_X}}|_\xi}^{-ad_X^*(\xi)} \mid X \in \mathfrak{g}\}$$

$$= \{-ad_X^*(\xi) \mid X \in \mathfrak{g}\}$$

$$= T_\xi(\mathcal{O}_\xi) = T_\xi(L)$$

where $\mathcal{O}_\xi$ denotes the coadjoint orbit through ξ.

$$\mathcal{O}_\xi = \{ Ad_g^*(\xi) \mid g \in G \}$$

Recall:

G : Lie gp. $\mathfrak{g} = \mathrm{Lie}(G)$

$\forall g \in G$,

$G \to G$, $x \mapsto g^{-1} x g$

$e \longmapsto e$.

$\rightsquigarrow$

$Ad_g : \mathfrak{g} \to \mathfrak{g}$

$Ad_{g^{-1}}$

$\updownarrow$

$- ad_x$

$\shortparallel$

$-[x, -]$

$\rightsquigarrow Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ $= (Ad_{g^{-1}})^*$

$\frac{d}{dt}\Big|_{t=0} Ad^*_{\exp(tx)}(\xi)$ G-action

$= ad_x^*(\xi)$ $\mathfrak{g}$-action

$\rightsquigarrow ad_x^* : \mathfrak{g}^* \to \mathfrak{g}^*$

$\Rightarrow L = \mathcal{O}_\xi$ #

<u>Cor</u>

Every coadjoint orbit of a Lie alg is a symplectic mfd.

<u>exer</u>

Consider $\mathfrak{g} = \mathfrak{so}(3)$.

$\mathfrak{g}^* \cong \mathbb{R}^3$ with

$\{x_1, x_2\} = x_3 \quad \{x_2, x_3\} = x_1,$

$\{x_3, x_1\} = x_2$

The symplectic leaves are

$$\{x_1^2 + x_2^2 + x_3^2 = r^2\}$$

In particular, S^2 is a symplectic mfd.

<u>Coisotropic submfd</u>

Let (M, Π) be a Poisson mfd,

S be a submfd of M.

Consider the conormal bundle of S

$$TS^\perp := \bigcup_{x \in S} \left\{ \xi \in T_x^*M \;\middle|\; \begin{array}{l} \langle \xi, v \rangle = 0 \\ \forall v \in T_x S \end{array} \right\}$$

A submfd S is coisotropic if

$$\Pi^{\#}(TS^{\perp}) \subseteq TS.$$

Lemma

The following are equivalent:

(i) S is coisotropic

(ii) $\forall f, g \in C^{\infty}(M)$ s.t. $f|_S = g|_S = 0$, we have $\{f, g\}|_S = 0$.
That is, the ideal
$$I(S) := \{ f \in C^{\infty}(M) \mid f|_S = 0 \}$$
is a Poisson subalg of $(C^{\infty}(M), \{,\})$.

(iii) $\forall f \in C^{\infty}(M)$ s.t. $f|_S = 0$, the Hamiltonian vector field X_f is tangent to S.

Remark

Suppose (M, Π) is symplectic.

Suppose

(i) If S is a coisotropic submfd of M, then $\pi^\#(TS^\perp) \subseteq TS$

$$\dim S \geq \frac{1}{2}\dim M$$

since $\pi^\# : T_p^*M \to T_pM$ is an iso, we have

$$\dim(\pi^\#(T_pS^\perp)) = \dim(T_pS^\perp)$$

$$= \dim M - \dim S$$

$$\wedge \qquad\qquad \wedge$$

$$\dim(T_pS) = \dim S$$

$$\Rightarrow \quad \dim S \geq \frac{1}{2}\dim M$$

(ii) If $\dim S = \frac{1}{2}\dim M$, then S is called a Lagrangian submanifold of M

Coisotropic submfd and Poisson maps are related in the following way:

Remark

Let $(M_1, \{\ ,\ \}_1)$ and $(M_2, \{\ ,\ \}_2)$ be Poisson mfd.

Then $M_1 \times M_2$ naturally carries a Poisson str:

Recall that

$$C^\infty(M_1) \otimes C^\infty(M_2) \hookrightarrow C^\infty(M_1 \times M_2)$$
$$f \otimes g \longmapsto pr_1^*(f) \cdot pr_2^*(g)$$

where

$$pr_1 : M_1 \times M_2 \to M_1, \quad (x,y) \mapsto x$$
$$pr_2 : M_1 \times M_2 \to M_2, \quad (x,y) \mapsto y$$

$f, f_1, f_2 \in C^\infty(M_1)$

$g, g_1, g_2 \in C^\infty(M_2)$

The Poisson bracket on $M_1 \times M_2$ is determined by

$$\{pr_1^*(f_1), pr_1^*(f_2)\} = pr_1^*(\{f_1, f_2\}_1)$$

$$\{pr_2^*(g_1), pr_2^*(g_2)\} = pr_2^*(\{g_1, g_2\}_2)$$

$$\{pr_1^*(f_1), pr_2^*(g_2)\} = 0$$

That is,

$$\{pr_1^*(f_1) \cdot pr_2^*(g_1), \ pr_1^*(f_2) \cdot pr_2^*(g_2)\}$$

$$= pr_1^*(\{f_1, f_2\}_1) \cdot pr_2^*(g_1 \cdot g_2)$$

$$+ pr_1^*(f_1 f_2) \cdot pr_2^*(\{g_1, g_2\}_2)$$

If the Poisson structures on M_1 and M_2 are locally represented by matrixes Π_1 and Π_2, respectively, then

$$\Pi_{M_1 \times M_2} = \begin{pmatrix} pr_1^* \Pi_1 & 0 \\ 0 & pr_2^* \Pi_2 \end{pmatrix}$$

The Poisson bivector of $M_1 \times M_2$ is often denoted by $\Pi_1 \oplus \Pi_2$

Now consider

$$M_1 \times \bar{M}_2$$

$$= M_1 \times M_2 \text{ with } \Pi_1 \oplus (-\Pi_2)$$

Prop

A map $\phi: M_1 \to M_2$ is a Poisson map
$\iff$ its graph

$$\Gamma_\phi = \{(x, \phi(x)) \in M_1 \times M_2 \mid x \in M_1\}$$

is a coisotropic submfd of $M_1 \times \bar{M}_2$

pf: exercise.

This proposition leads to the following:

Def

(i) A relation $R: M_1 \to M_2$ is a submfd $M_1 \times \bar{M}_2$

(ii) A Poisson relation is a coisotropic submfd of $M_1 \times \bar{M}_2$.

Remark

① $\Phi : M_1 \to M_2$ is a Poisson map

$\Leftrightarrow$ Γ_Φ is a Poisson relation from M_1 to M_2

② Any coisotropic submfd of M_2 can be viewed as a Poisson relation

$$R : M_0 = \{x_0\} \longrightarrow M_2$$

Def

For $R_1 : M_1 \to M_2$, $R_2 : M_2 \to M_3$,

$$R_2 \circ R_1 := \left\{ (x_1, x_3) \in M_1 \times M_3 \;\middle|\; \begin{array}{l} \exists\, x_2 \in M_2 \text{ s.t.} \\ (x_1, x_2) \in R_1 \\ (x_2, x_3) \in R_2 \end{array} \right\}$$

If $\Phi_1 : M_1 \to M_2$, $\Phi_2 : M_2 \to M_3$ are Poisson maps

then

$$\Gamma_{\Phi_2} \circ \Gamma_{\Phi_1} = \left\{ (x_1, x_3) \;\middle|\; \begin{array}{l} \exists\, x_2 \in M_2 \\ (x_1, x_2) \in \Gamma_{\Phi_1} \Leftrightarrow x_2 = \Phi_1(x_1) \\ (x_2, x_3) \in \Gamma_{\Phi_2} \end{array} \right\}$$

$$\Leftrightarrow x_3 = \Phi_2(x_2) \qquad \Downarrow \qquad x_3 = \Phi_2(\Phi_1(x_1))$$

$$\parallel$$

$$\Gamma_{\Phi_2 \circ \Phi_1}$$

Remark

necessarily

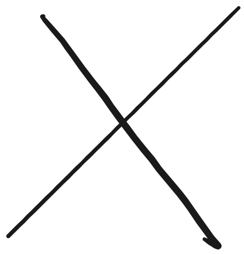
$R_1 \circ R_2$ is NOT a submfd.

e.g. $R_1 = \{x^2+y^2=1\} \subseteq \overset{M_1}{\overset{\shortparallel}{\mathbb{R}}} \times \overset{M_2}{\overset{\shortparallel}{\mathbb{R}}}$

$R_2 = \{x^2+y^2=1\} \subseteq \overset{M_2}{\overset{\shortparallel}{\mathbb{R}}} \times \overset{M_3}{\overset{\shortparallel}{\mathbb{R}}}$

$\Rightarrow R_2 \circ R_1 = \{(x, z) \in \mathbb{R}^2 \mid \exists y \text{ s.t. } x^2+y^2=1,\ y^2+z^2=1\}$

$\Leftrightarrow x^2 = z^2$

$=$ 

is NOT a mfd.