Geometric models for noncommutative algebras

Fall 2025, week 1

Introduction

Two main directions in this course:

(I)

Symmetries

geometry	algebra
Lie group G	Lie algebra $\mathfrak{g}$
left invariant vector fields on G or T_eG first-order differential operators $\longleftrightarrow$	vec. sp. with bilinear map $[\ ,\ ] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ s.t. ....
left invariant differential operators on G	universal enveloping algebra $\mathcal{U}(\mathfrak{g})$

We also can weaker symmetries:

Lie groupoid $\mathcal{G}$	Lie algebroid A

left invariant vector fields on G	vector bundle A with a Lie bracket on $\Gamma(A)$ s.t. .... global sections
left invariant differential operators on G	universal enveloping algebra $\mathcal{U}(A)$ of A

(II)

spaces and algebra

geometry	algebra
manifold M	function alg $C^\infty(M)$
smooth map $N \to M$	alg morphism $C^\infty(M) \to C^\infty(N)$
vector field on M	derivation of $C^\infty(M)$ $X: A \xrightarrow{\text{linear}} A$, $X(ab) = X(a)b + aX(b)$
vector bundle over M	projective $C^\infty(M)$-module
Lie algebroid A In particular, Lie alg $\mathfrak{g}$	Chevalley-Eilenberg dg alg. $(\Gamma(\wedge^\bullet A^*), d_{CE})$ ref: $(\wedge^\bullet \mathfrak{g}^*, d_{CE})$ search for Lie alg cohomology
differential graded manifold $(\mathcal{M}, Q)$	dg alg. $(C^\infty(\mathcal{M}), Q)$

functions on $(\mathcal{M}, Q)$

Lie algebras and universal enveloping alg

Preliminaries: tensor product

Let V, W be vector spaces over a field $\Bbbk$. The tensor product is the vector space

$$V \otimes W = \Bbbk^{(V \times W)} \Big/ R$$

where

$\Bbbk^{(V \times W)}$ = the vector space freely generated by the set $V \times W$

$$= \left\{ \sum_{(x,y) \in V \times W} a_{(x,y)} \cdot (x,y) \;\middle|\; \begin{array}{l} a_{(x,y)} \in \Bbbk \\ a_{(x,y)} = 0 \\ \text{except finite} \\ (x,y) \in V \times W \end{array} \right\}$$

and R is the subspace of $\Bbbk^{(V \times W)}$ spanned by

$(x_1+x_2, y) - (x_1, y) - (x_2, y),$ $\quad x, x_1, x_2 \in V$

$(x, y_1+y_2) - (x, y_1) - (x, y_2),$ $\quad y, y_1, y_2 \in W$

$(c\cdot x, y) - c\cdot(x,y)$ $\quad c \in \mathbb{k}$

$(x, c\cdot y) - c\cdot(x,y)$

Remark

① An arbitray element in $V\otimes W$ is of the form

$$\sum_{i=1}^{k} x_i \otimes y_i = [(x_i, y_i)] \in \mathbb{k}^{(V\times W)}/R$$

② Furthermore,

$$(ax_1+bx_2)\otimes y = a\cdot x_1\otimes y + b\cdot x_2\otimes y$$

$$x\otimes(ay_1+by_2) = a\cdot x\otimes y_1 + b\cdot x\otimes y_2$$

$$(a\cdot x)\otimes y = a\cdot(x\otimes y) = x\otimes(a\cdot y)$$

③ If $v_1, \cdots, v_n$ form a basis for V

$w_1, \cdots, w_m$ " basis for W,

then

$v_1\otimes w_1, \quad v_1\otimes w_2, \cdots, \quad v_1\otimes w_m,$

$v_2\otimes w_1, \quad \cdots, \quad v_2\otimes w_m$

$\vdots \qquad\qquad \vdots$

$v_n \otimes w_1, \ldots \quad \cdots \quad , v_n \otimes w_m$

form a basis $V \otimes W$.

④ $\left\{ \begin{array}{c} \text{finite-dimensional} \\ \text{vector spaces} \end{array} \right\} \big/ \text{isomorphism} \longleftrightarrow \mathbb{Z}_{\geq 0}$

$$V \longleftrightarrow \dim V$$

For $m, n \in \mathbb{Z}_{\geq 0}$,

$$V \oplus W \longleftrightarrow n+m$$

$\dim(V \oplus W) = \dim(V) + \dim(W)$

$$V \otimes W \longleftrightarrow n \cdot m$$

$\dim(V \otimes W) = \dim(V) \cdot \dim(W)$

Def

Let $V_1, \cdots, V_n$ be vector spaces over $\mathbb{k}$.

$$V_1 \otimes V_2 \otimes \cdots \otimes V_n \overset{\text{def.}}{=} \mathbb{k}^{(V_1 \times V_2 \times \cdots \times V_n)} \big/ R$$

where

$$R = \operatorname{span}\left\{ \begin{array}{l} (v_1, v_2, \cdots, v_i + v_i', \cdots, v_n) \\ \quad - (v_1, \cdots, v_i, \cdots, v_n) - (v_1, \cdots, v_i', \cdots, v_n), \\ (v_1, \cdots, c \cdot v_i, \cdots, v_n) \\ \quad - c \cdot (v_1, \cdots, v_i, \cdots, v_n) \\ v_i \in V_i, \ i = 1, \cdots, n \end{array} \right\}$$

The tensor algebra generated by a vector space V is

$$TV = \bigoplus_{n=0}^{\infty} \underbrace{\overbrace{V \otimes V \otimes \cdots \otimes V}^{n \text{ times}}}_{= V^{\otimes n}}$$

vector space

$\Rightarrow$ we have scalar product

and addition

which is equipped with the multiplication

$$\underset{\in\, V^{\otimes i} \subseteq \mathbb{T}V}{(v_1 \otimes \cdots \otimes v_i)} \cdot \underset{\in\, V^{\otimes j} \subseteq \mathbb{T}V}{(v_{i+1} \otimes \cdots \otimes v_{i+j})}$$

$$= \underset{\in\, V^{\otimes (i+j)} \subseteq \mathbb{T}V}{v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_{i+j}}$$

Lie algebra

Suppose $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$ for convenience.

Def.

A Lie algebra is a vector space $\mathfrak{g}$ together with bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the Lie bracket, satisfying

(i) anticommutativity:

$$[x, y] = -[y, x]$$

(ii) Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

$\forall \; x, y, z \in \mathfrak{g}$.

<u>Remark</u>

① The binary operation $[\, , \,]$ is almost never <u>associative</u>

" $[x, [y, z]] = [[x, y], z]$ "

$x \cdot (y z)$ $\quad$ $(x \cdot y) \cdot z$

② Jacobi identity $\Longleftrightarrow$

$[x, -]$ is a derivation $\forall \, x \in \mathfrak{g}$,

i.e.

exer

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

<u>Example</u> Let A be an associative alg.

① $(A, [\,,\,])$, $[a,b] := a\cdot b - b\cdot a$, ← Commutator

is a Lie algebra (exercise)

(1.5) Since $\mathrm{End}_k A = \{f : A \to A,\ f \text{ is linear}\}$

is an alg. w.r.t. Composition $\circ$,

it can be regarded as a Lie alg. by ①

② $\mathrm{Der}(A) := \{\text{derivations of } A\}$

$$= \left\{ f \in \mathrm{End}(A) \;\middle|\; \begin{array}{l} f(ab) = f(a)\,b \\ \quad + a\,f(b) \end{array} \right\}$$

is a Lie subalgebra of $\mathrm{End}(A)$.

To show this claim, check:

$\forall f, g \in \mathrm{Der}(A)$,

$$[f,g] = f\circ g - g\circ f$$

is still in $\mathrm{Der}(A)$, i.e.

$$[f,g](ab) \overset{\text{check}}{=} [f,g](a)\cdot b \qquad \text{exer}$$

$+ a \cdot [f,g](b))$

③ $\mathfrak{sl}_2(\mathbb{C}) = \{X \in M_2(\mathbb{C}) \mid tr(X) = 0\}$

dim $\mathfrak{sl}_2(\mathbb{C}) = 3$

In fact,

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis for $\mathfrak{sl}_2(\mathbb{C})$.

The Lie bracket is determined by

$$[E,F] = E \cdot F - F \cdot E$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H$$

exer

Prove or disprove:

dim $\mathfrak{g} = 2 \Rightarrow [X,Y] = 0 \quad \forall X, Y \in \mathfrak{g}$

$$[H,F] = -2F$$

$$[H, E] = 2E.$$

$[H, E] = -2$

__Def__

Let $\mathfrak{g}$ be a Lie alg.

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra

$$\mathcal{U}(\mathfrak{g}) = \mathbb{T}\mathfrak{g} \Big/ \underbrace{\left\langle x \otimes y - y \otimes x - [x, y] \right\rangle_{x, y \in \mathfrak{g}}}$$

the ideal generated by all the elements of the form $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{g}$

__Remark__

$\mathfrak{g}$ can be considered as a subspace of $\mathcal{U}(\mathfrak{g})$:

$$\mathfrak{g} = \mathfrak{g}^{\otimes 1} \hookrightarrow \mathbb{T}\mathfrak{g} \twoheadrightarrow \mathcal{U}(\mathfrak{g})$$

Furthermore,

$[u,v] = u \cdot v - v \cdot u$

$(\mathfrak{g}, [\,,\,]) \subset (\mathcal{U}(\mathfrak{g}), \underset{\sim}{[\,,\,]})$ is a Lie subalgebra of

since, by the construction of $\mathcal{U}(\mathfrak{g})$,

$$\underset{\in \mathfrak{g}}{x} \cdot \underset{\in \mathfrak{g}}{y} - y \cdot x = [x,y] \quad \leftarrow \text{in } \mathcal{U}(\mathfrak{g})$$

<u>Prop</u> (Universal property for $\mathcal{U}(\mathfrak{g})$)

Let $\mathfrak{g}$ be a Lie alg (over $\mathbb{k}$), A be an associative alg (over $\mathbb{k}$).

If f : $\mathfrak{g} \to A$ is a linear map s.t.

Lie alg morphism

$$f([x,y]) = \underbrace{f(x) \cdot f(y) - f(y) \cdot f(x)}_{[f(x),\, f(y)] \text{ in } A}$$

then $\exists!$ algebra morphism

$$\tilde{f} : \mathcal{U}(\mathfrak{g}) \longrightarrow A$$

s.t. $\tilde{f} \circ \varphi = f$, where φ is the map

$$\begin{tikzcd} \mathfrak{g} \arrow[r, hook] \arrow[rr, bend left, "\varphi"] & T\mathfrak{g} \arrow[r, two heads] & \mathcal{U}(\mathfrak{g}) \end{tikzcd}$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi"] \arrow[dr, "\forall f"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & A \end{tikzcd}$$

$\forall f$ Lie alg mor, $\exists!\, \tilde{f}$ alg mor

exer: prove the proposition

Remark

① A representation of a Lie alg $\mathfrak{g}$ is a vector space V together with a Lie alg mor

comutator

$$\phi : \mathfrak{g} \to \mathrm{End}(V)$$

idea: V has a $\mathfrak{g}$-symmetry

e.g. For $\mathfrak{sl}_2(\mathbb{C})$, consider $V = \mathbb{C}^2$

$$\phi : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}(\mathbb{C}^2),$$

$X \in \mathfrak{sl}_2(\mathbb{C})$, $\phi(X)(\vec{v}) = X \cdot \vec{v}$

② A representation of an associative alg. A

is a vector space V together with an alg mor

composition

$$\Phi : A \longrightarrow \mathrm{End}(V)$$

$\Leftrightarrow$ V is a module over A.

③ By the universal property,

a representation of $\mathfrak{g}$ (as a Lie alg.)

$\Leftrightarrow$

a " of $\mathcal{U}\mathfrak{g}$) (as an alg.)

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, hook] \arrow[dr, "\phi"'] & \mathcal{U}\mathfrak{g} \arrow[d, dashed, "\exists!\,\Phi"] \\
& \mathrm{End}(V)
\end{tikzcd}$$

Lie alg representation

Lie alg mor.

alg mor

alg representation

$\mathcal{U}\mathfrak{g}$-module

Matrix groups and their Lie algebras

(and actions)

In general, we use groups to study symmetries.

Example

Consider $\mathbb{Z}_2 = \{\pm 1\} \subseteq \mathbb{R}$

A function $f : \mathbb{R} \to \mathbb{R}$ is even iff f is "$\mathbb{Z}_2$-invarint", i.e.

$$f(g \cdot x) = f(x) \quad \forall g \in \mathbb{Z}_2$$

f is odd $\iff$ f is "$\mathbb{Z}_2$-equivarint"

i.e.

$$f(g \cdot x) = g \cdot f(x) \quad \forall g \in \mathbb{Z}_2$$

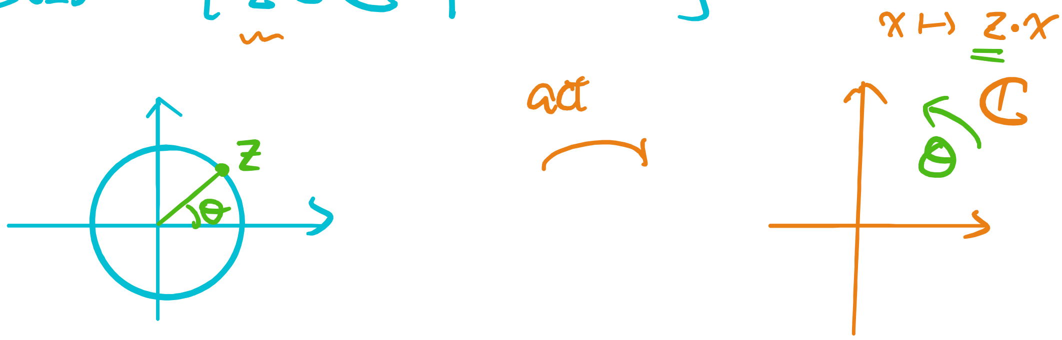
some symmetries

So evenness and oddness of functions can be understood by the $\mathbb{Z}_2$-action on $\mathbb{R}$.

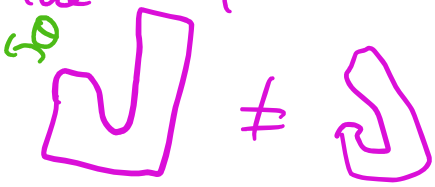
$\mathbb{Z}_2$ (or finite group) $\longleftrightarrow$ study of discrete symmetries

Lie group $\longleftrightarrow$ study of smooth symmetries

$U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$



NOTE: The space



doesn't have a "nice" U(1)-action

A big success in Lie theory is:

Lie groups $\overset{\text{NOT 1-1}}{\longleftrightarrow}$ Lie algebras

but almost

very difficult

linear algebra much easier

To avoid differential geometry, let's consider "matrix groups".

Here a matrix group is a closed subgroup of $GL_n(\mathbb{C})$.

$$GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) \neq 0\} = \{\text{invertible } n\times n \text{ matrices}\}$$

Consider $M_n(\mathbb{C})$ as $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$

metric space

we have "open", "closed", ...

Lemma

$$\det : M_n(\mathbb{C}) \to \mathbb{C}$$

exer: prove the lemma.

is a smooth function (in particular, a continuous function)

$\mathbb{C} \cong \mathbb{R}^2$

$\cup$ open

Therefore,

$$GL_n(\mathbb{C}) = \det^{-1}(\mathbb{C} \setminus \{0\})$$

is an open subset of $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$

In particular, $GL_n(\mathbb{C})$ is a manifold.
(in particular, a topological space)

We say G is a closed subgroup of $GL_n(\mathbb{C})$ if

(i) G is a closed subset of $GL_n(\mathbb{C})$

(ii) G is a subgroup of $GL_n(\mathbb{C})$.

in general, a Lie group

Thm (Closed subgroup thm).
If H is a closed subgroup of $GL_n(\mathbb{C})$ then H is an embedded Lie subgroup of G.

In particular, the maps

$$H \times H \to H, \quad (x,y) \mapsto x \cdot y$$

$$H \to H, \qquad x \longmapsto x$$

are smooth.