Calculus — Homework 9 (Fall 2025)

1. Prove that for all real numbers x and y
   (a) $|\cos x - \cos y| \leq |x - y|$.
   (b) $|\sin x - \sin y| \leq |x - y|$.

2. Show that
$$\tan x > x,$$
for all x in $(0, \frac{\pi}{2})$.

3. True or false? Explain your answers.
   (a) The function $f(x) = x^2$ is an increasing function on $(-\infty, \infty)$.
   (b) The function $f(x) = x^2$ is a decreasing function on $(-\infty, \infty)$.
   (c) The function $f(x) = x^2$ is an increasing function on $(0, \infty)$.
   (d) The function $f(x) = x^3$ is an increasing function on $(-\infty, \infty)$.

4. Suppose a function f has derivative
$$f'(x) = x^3(x-1)^2(x+1)(x-2).$$
At what numbers x, if any, does f have a local maximum? A local minimum?

5. Find the critical points, local maximums and local minimums of f.
   (a) $f(x) = x^3 - 3x + 2$.
   (b) $f(x) = x + \frac{1}{x}$.
   (c) $f(x) = |x^2 - 5|$.
   (d) $f(x) = x - \cos x$.

6. Find the critical points. Then find and classify all the extreme values.
   (a) $f(x) = x^2 - 4x + 1, \quad 0 \leq x \leq 3$.
   (b) $f(x) = \dfrac{x^2}{1+x^2}, \quad -1 \leq x \leq 2$.
   (c) $f(x) = \sin 2x - x, \quad 0 \leq x \leq \pi$.
   (d) $f(x) = 1 - \sqrt[3]{x-1}, \quad x \in (-\infty, \infty)$.
   (e) $f(x) = \begin{cases} x^2 + 2x + 2, & x < 0, \\ x^2 - 2x + 2, & 0 \leq x \leq 2. \end{cases}$

7. Describe the concavity of the graph and find the points of inflection (if any).
   (a) $f(x) = x + \dfrac{1}{x}$.
   (b) $f(x) = x^3(1-x)$.
   (c) $f(x) = \sin^2 x, \quad 0 < x < \pi$.

8. Suppose $f \in C^2[0,1]$ such that $f(0) = f(1) = 0$ and $f''(x) < 0$ for all $x \in (0,1)$. Prove that $f(x) > 0$ for all $x \in (0,1)$.

9. Let $f \in C^2(a,b)$.
   (a) Let $x_0 \in (a,b)$ and $h > 0$ such that $x_0 - h, x_0 + h \in (a,b)$. Prove that there exists $\xi \in (x_0 - h, x_0 + h)$ such that
$$f(x_0 + h) - 2f(x_0) + f(x_0 - h) = f''(\xi)h^2.$$
(Hint: Consider the function $f(x) - q(x)$, where $q(x) = a(x-x_0)^2 + b(x-x_0) + c$ is the quadratic polynomial satisfying
$$q(x_0) = f(x_0), \qquad q(x_0 - h) = f(x_0 - h), \qquad q(x_0 + h) = f(x_0 + h).$$
Apply the mean value theorem to it twice.)

(b) Suppose for any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, we have

$$f\Big(\frac{x_1 + x_2}{2}\Big) \le \frac{1}{2}\Big(f(x_1) + f(x_2)\Big).$$

Prove that $f''(x) \ge 0$ for all $x \in (a, b)$.

10. Suppose $f \in C^2(a, b)$, $f''(x) \le 0$ for all $x \in (a, b)$, and $c \in (a, b)$. Prove that

$$f(c) + f'(c) \cdot (x - c) \ge f(x)$$

for all $x \in (a, b)$.