Calculus — Homework 8 (Fall 2025)

1. Differentiate the following functions.

   (a) $f(x) = (1 - 2x)^{-1}$.
   (b) $f(x) = (1 + 2x)^5$.
   (c) $f(x) = \left(x - \frac{1}{x}\right)^4$.
   (d) $f(x) = 3\cos x - 4\sec x$.
   (e) $f(x) = \sin^2 x$.
   (f) $f(x) = \tan(x^2)$.
   (g) $f(x) = \cos(\sqrt{x})$, $x > 0$.
   (h) $f(x) = \sqrt[3]{\frac{x}{1+x^2}}$, $x > 0$.
   (i) $f(x) = \sqrt{\sin x \cos x}$, $0 < x < \pi/2$.
   (j) $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$, $x > 0$.
   (k) $f(x) = (x+1)^{1/3}(x+2)^{2/3}$.

2. Express dy/dx in terms of x and y.

   (a) $x^2 + y^2 = 4$.
   (b) $x^3 + y^3 - 3xy = 0$.
   (c) $\sin(x + y) = xy$.
   (d) $\sqrt{x} + \sqrt{y} = 4$, $x, y > 0$.

3. Find equations for the tangent line at the point indicated.
   (a) $9x^2 + 4y^2 = 72$; $(2, 3)$.
   (b) $x^2 + xy + 2y^2 = 28$; $(-2, -3)$.
   (c) $x = \cos y$; $(\frac{1}{2}, \frac{\pi}{3})$.

4. Let
$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$$
   Is f differentiable at $x = 0$? Is f twice differentiable (i.e. f' is differentiable) at $x = 0$? Explain your answers.

5. Prove that if $|f(x) - (3x + 2)| \leq |x|^{\frac{3}{2}}$ for any real numbers x, then f is differentiable at 0, and $y = 3x + 2$ is the tangent line of the graph of f at $(0, f(0))$. (Approximation of f by a line around $x = 0$. Recall that $0^a = 0$ for any $a \neq 0$.)

6. Find the smallest positive integer n such that $\frac{d^n}{dx^n}(x^{10}\sin x)\Big|_{x=0} \neq 0$ and find this value. (Hint: derive a formula of $\frac{d^n}{dx^n}(fg)$ from the product rule.)

7. Suppose $f, g \in C[0, 1]$, f and g are differentiable on $(0, 1)$, $f(0) = g(0)$, and $f'(x) < g'(x)$ for all $x \in (0, 1)$. Prove that $f(1) < g(1)$.

8. Suppose $f \in C^1[0, 1]$, $f(x) \in [0, 1]$, and $|f'(x)| < 1$ for all $x \in [0, 1]$.
   (a) Prove that there exists a constant M, $0 \leq M < 1$, such that $|f'(x)| \leq M$ for all $x \in [0, 1]$.
   (b) Let M be a number such that $|f'(x)| \leq M$ for all $x \in [0, 1]$. Prove that $|f(x) - f(y)| \leq M|x - y|$.
   (c) Let $x_0 \in [0, 1]$. Since $f([0, 1]) \subset [0, 1]$, we can define a sequence $(x_n)_{n=1}^{\infty}$ in $[0, 1]$ by iteration:
$$x_1 = f(x_0),\ x_2 = f(x_1),\ \cdots,\ x_{n+1} = f(x_n),\ \cdots.$$
   Prove that the sequence $(x_n)_{n=1}^{\infty}$ is convergent, and
$$f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} x_n.$$

9. Suppose $f \in C^n(a, b)$. Assume there exists $a < x_0 < x_1 < \cdots < x_n < b$ such that $f(x_0) = f(x_1) = \cdots = f(x_n) = 0$. Prove that there exists $\xi \in (x_0, x_n)$ such that $f^{(n)}(\xi) = 0$.

10. The polynomial
$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$
   is called the **Legendre polynomial** of degree n. Prove that $P_n(x)$ has n distinct real zeros in $(-1, 1)$.

11. Find the polynomial $C_n(t)$ of degree n such that

$$C_n(\cos x) = \cos nx.$$

12. Find the polynomial $T_n(t)$ of degree n such that

$$T_n(\cos x) \cdot \sin x = \sin(n+1)x.$$