Calculus — Homework 7 (Fall 2025)

1. Let $f(x)$ be a polynomial of degree n. Suppose that
$$f(-x) = -f(x), \qquad \forall x \in [-1, 1].$$
Prove that
   (a) $f(-1000) = -f(1000)$;
   (b) n is an odd number.

2. True or false? Explain how each of your answers is consistent with the extreme value theorem.
   (a) The function $f(x) = x^2$ attains a maximum value on $[-1, 1]$.
   (b) The function $f(x) = x^2$ attains a minimum value on $[-1, 1]$.
   (c) The function $f(x) = x^2$ attains a maximum value on $(-1, 1)$.
   (d) The function $f(x) = x^2$ attains a minimum value on $(-1, 1)$.
   (e) The function $f(x) = x^2$ is bounded on $(-1, 1)$.

3. Write an equation for the tangent line at $(c, f(c))$.
   (a) $f(x) = x^2 - 4x, \quad c = 3.$
   (b) $f(x) = \sqrt{x}, \quad c = 1.$

4. Draw the graph of f; indicate where f is not differentiable, and indicate where f is not continuous.
   (a) $f(x) = \sqrt{|x|}$.
   (b) $f(x) = |x^2 - 4|$.
   (c) $f(x) = \begin{cases} x^2, & |x| \le 1, \\ 2 - x, & |x| > 1. \end{cases}$

5. Differentiate the following functions.
   (a) $f(x) = 1 - x$.
   (b) $f(x) = 11x^5 - 6x^3 + 8$.
   (c) $f(x) = \dfrac{3}{x^2}$.
   (d) $f(x) = (x^2 - 1)(x - 3)$.
   (e) $f(x) = \dfrac{x^3}{1 - x}$.
   (f) $f(x) = \Big(1 + \dfrac{1}{x}\Big)\Big(1 + \dfrac{1}{x^2}\Big)$.

6. Find the point(s) where the tangent line is horizontal.
   (a) $f(x) = (x - 2)(x^2 - x - 11)$.
   (b) $f(x) = x^2 - \dfrac{16}{x}$.

7. Let $f_1, \cdots, f_n$ be functions that are differentiable at x. Prove that for any $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$,
$$(\alpha_1 f_1 + \cdots + \alpha_n f_n)'(x) = \alpha_1 \cdot f_1'(x) + \cdots + \alpha_n \cdot f_n'(x).$$

8. Let $f(x) = x + x^2 + \cdots + x^n$.
   (a) Show that $f(x) = \dfrac{x - x^{n+1}}{1 - x}$ for all $x \neq 1$.
   (b) Find the value of the sum $\displaystyle\sum_{j=1}^{n} \frac{j}{2^{j-1}}$. (Hint: Consider $f'(x)$.)
   (c) Find the value of the infinite sum $\displaystyle\sum_{j=1}^{\infty} \frac{j}{2^{j-1}}$, which is, by definition, the limit $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} \frac{j}{2^{j-1}}$.

9. Let $f(x)$ be a polynomial of degree n with zeros

$$a_1 < a_2 < \cdots < a_n.$$

Let $g(x)$ be a polynomial of degree at most $n-1$. Suppose that $A_1, \ldots, A_n \in \mathbb{R}$ are numbers such that

$$\frac{g(x)}{f(x)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_n}{x-a_n}, \qquad \forall x \in \mathbb{R} \setminus \{a_1, \cdots, a_n\}.$$

(a) Prove that $A_j = \dfrac{g(a_j)}{f'(a_j)}$.

(b) Assume, in addition, that $A_k A_{k+1} > 0$ for $k = 1, \cdots, n-1$. Prove that $g(x)$ is of degree $n-1$, and it has $n-1$ distinct real zeros.

10. Let $f(x)$ and $q(x)$ be polynomials such that

$$f(x) = (x-a)^2 q(x), \qquad \forall x \in \mathbb{R}.$$

Prove that $q(a) = \dfrac{f''(a)}{2!}$.