Calculus — Homework 6 (Fall 2025)

1. For $x \in \mathbb{R}$, let $[x]$ denote its Gauss symbol, i.e. the largest integer which is less than or equal to x. In the interval $(-\infty, \infty)$, define the following functions:
   (a) $f(x) = x - [x]$,
   (b) $g(x) = x[x]$,
   (c) $h(x) = [x] \sin x$.

   Find the points of discontinuity of these three functions.

2. Suppose $f(x)$ is a continuous function on $[0, 1]$. Assume that for any positive integers p, q with $p < q$, we have
$$f(\frac{p}{q}) = \frac{2p}{q}.$$
   Prove that $f(x) = 2x$ for all $x \in [0, 1]$.

3. Evaluate the limits.
   (a) $\lim_{x\to\pi} \sin(x - \sin x)$.
   (b) $\lim_{x\to 0} \sin\left(\frac{\pi}{2}\cos(\tan x)\right)$.
   (c) $\lim_{x\to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3\sec(2x)}}\right)$.
   (d) $\lim_{x\to 0^+} \sin\left(\frac{\pi}{2}\cos(\sqrt{x})\right)$.

4. Use the intermediate value theorem to show that there is a solution of the given equation in the indicated interval.
   (a) $2x^3 - 4x^2 + 5x - 4 = 0, \quad [1, 2]$.
   (b) $\sin x + 2\cos x - x^2 = 0, \quad [0, \pi/2]$.

5. (Brouwer fixed-point theorem.) Show that if f is continuous on $[0, 1]$ and $0 \le f(x) \le 1$ for all x in $[0, 1]$, then there exists at least one point c in $[0, 1]$ at which $f(c) = c$. (HINT: Apply the intermediate value theorem to the function $g(x) = x - f(x)$.)

6. Let $f(x)$ and $g(x)$ be two continuous functions defined in $[0, 1]$. Suppose that $f(0) < g(0)$ and $f(1) > g(1)$.
   (a) Prove that there exists $x_0 \in (0, 1)$ such that $f(x_0) = g(x_0)$.
   (b) Suppose, in addition, that $(f(x))^2 = (g(x))^2$ for any $x \in [0, 1]$. Prove that there exists $x_1 \in (0, 1)$ such that $f(x_1) = g(x_1) = 0$.

7. Let $f(x)$ and $g(x)$ be tw continuous functions defined in $[0, 1]$. Suppose that
   (i) $f(x) \ne 0$ for all $x \in [0, 1]$,
   (ii) $(f(x))^2 = (g(x))^2$ for all $x \in [0, 1]$,
   (iii) $f(\frac{1}{2})g(\frac{1}{2}) < 0$.

   Prove that $f(x) = -g(x)$ for all $x \in [0, 1]$.

8. Let $f(x)$ be a real valued function which is continuous in $(-\infty, \infty)$. Suppose that
$$\lim_{n\to\infty} \frac{f(n\pi + \frac{\pi}{3})}{\sin(n\pi + \frac{\pi}{3})} = 1.$$
   Prove that $f(x)$ has infinitely many zeros in the half line $(0, +\infty)$.