Calculus — Homework 5 (Fall 2025)

1. Suppose that
$$\lim_{x\to c} g(x) = d \qquad \text{and} \qquad \lim_{y\to d} f(y) = L.$$
Prove that if there exists $p > 0$ such that $g(x) \neq d$ for any $0 < |x - c| < p$, then
$$\lim_{x\to c} f(g(x)) = L.$$

2. Suppose $f(x)$ is defined in the set $S = \{x \in \mathbb{R} \mid 0 < |x-2| < 2\}$, and $\lim\limits_{x\to 2} \dfrac{f(x)-3}{x-2} = 1$.
   (a) Prove that there exists $0 < \delta < 2$ such that if $0 < |x-2| < \delta$, then $|f(x) - 3| < 2|x-2|$.
   (b) Prove that $\lim\limits_{x\to 2} f(x)$ exists, and find its value.
   (c) Does the limit $\lim\limits_{x\to 2} \dfrac{(f(x))^2 - \frac{3xf(x)}{2}}{x-2}$ exist? If it exists, find its value.

3. Assume that m and n are positive integers. Evaluate the limits that do exist. If the limit does not exist, explain why.
   (a) $\lim\limits_{x\to 1}\left(\dfrac{x^m - 1}{x^n - 1}\right)$.
   (b) $\lim\limits_{x\to 1}\left(\dfrac{m}{1-x^m} - \dfrac{n}{1-x^n}\right)$.
   (c) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$.
   (d) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{\sin 5x}$.
   (e) $\lim\limits_{x\to 0} \dfrac{\tan^2(3x)}{4x^2}$.
   (f) $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{2x}$.
   (g) $\lim\limits_{x\to \pi/4} \dfrac{\sin x}{x}$.
   (h) $\lim\limits_{x\to \pi} \dfrac{\sin x}{x-\pi}$.
   (i) $\lim\limits_{x\to 0} \dfrac{\sin 7x - \sin 5x}{\sin x}$.

4. State the precise definition of $\lim\limits_{x\to -\infty} f(x) = L$, and use it to prove that $\lim\limits_{x\to -\infty} \dfrac{1}{x} = 0$.

5. Evaluate the limit $\lim\limits_{x\to 0} |x| \cos(\dfrac{1}{x^2})$. Prove your answer.

6. Evaluate the limits.
   (a) $\lim\limits_{x\to\infty} \sqrt{\dfrac{8x^2-3}{2x^2+x}}$.
   (b) $\lim\limits_{x\to -\infty} \left(\dfrac{1-x^3}{x^2+7x}\right)^5$.
   (c) $\lim\limits_{x\to -\infty} \dfrac{\sqrt{x^2+1}}{x+1}$.
   (d) $\lim\limits_{x\to\infty} \dfrac{2x^{5/3} - x^{1/3} + 7}{x^{9/5} + 3x + \sqrt{x}}$.

7. Determine whether the function is continuous at the indicated point. Explain your answers.
   (a) $f(x) = x^3 - 5x + 1$, $x = 2$.
   (b) $f(x) = x\sin x + \cos^2 x$, $x = 1$.
   (c) $f(x) = \tan x$, $x = \pi/2$.
   (d) $f(x) = \sqrt{(x-1)^2 + 5}$, $x = 1$.
   (e) $f(x) = |4 - x^2|$, $x = 2$.
   (f) $f(x) = \begin{cases} x^2 + 4, & x < 2, \\ x^3, & x \geq 2, \end{cases}$ $x = 2$.
   (g) $f(x) = \begin{cases} x^2 + 5, & x < 2, \\ x^3, & x \geq 2, \end{cases}$ $x = 2$.
   (h) $f(x) = \begin{cases} \dfrac{|x-1|}{x-1}, & x \neq 1, \\ 0, & x = 1, \end{cases}$ $x = 1$.
   (i) $f(x) = \begin{cases} -x^2, & x < 0, \\ 0, & x = 0, \\ 1/x^2, & x > 0, \end{cases}$ $x = 0$.