# Calculus — Homework 4 (Fall 2025)

1. Use the $\varepsilon$-$\delta$ argument (i.e. the precise definition of limit) to prove that: if $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$ and $\lim_{x\to c} h(x) = N$, then
$$\lim_{x\to c}\Big(3f(x) + 4g(x) - 5h(x)\Big) = 3L + 4M - 5N.$$

2. Let $f$ be a function defined on $(c-p, c+p)\setminus\{c\}$, where $p > 0$. Assume that $L \in \mathbb{R}$ is a number with the property: for any sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in (c-p, c+p)\setminus\{c\}$ and $\lim_{n\to\infty} x_n = c$, we have $\lim_{n\to\infty} f(x_n) = L$. Prove that $\lim_{x\to c} f(x) = L$.

3. Prove that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$.

4. Suppose that $\lim_{x\to c} f(x)$ exists and $k \in \mathbb{R}$. Prove that $\lim_{x\to c} (k \cdot f(x)) = k \cdot \lim_{x\to c} f(x)$.

5. Suppose that $c > 0$ and $n \in \mathbb{N}$. Prove that $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$.

6. Decide whether or not the indicated limit exists. Evaluate the limits that do exist. If the limit does not exist, explain why.

   (a) $\lim_{x\to -1} |x|(x^4 - 3)$.

   (b) $\lim_{x\to 1} \dfrac{x}{x+1}$.

   (c) $\lim_{x\to -1} \dfrac{1-x}{x+1}$.

   (d) $\lim_{x\to 0} \dfrac{x(x+1)}{2x^2}$.

   (e) $\lim_{x\to 1} \dfrac{x}{|x|}$.

   (f) $\lim_{x\to 1^+} \dfrac{\sqrt{x-1}}{x}$.

   (g) $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$.

   (h) $\lim_{x\to -1^+} x^3(x^4+1)$.

   (i) $\lim_{x\to 1} \dfrac{x^{10}-1}{x^3-1}$.

   (j) $\lim_{x\to -4} \Big(\dfrac{2x}{x+4} + \dfrac{8}{x+4}\Big)$.

   (k) $\lim_{h\to 0} h\Big(1 - \dfrac{1}{h}\Big)$.

   (l) $\lim_{h\to 0} \dfrac{1 - 1/h^2}{1 + 1/h^2}$.

   (m) $\lim_{x\to 2^+} f(x)$ if $f(x) = \begin{cases} 2x-1, & x \le 2 \\ x^2 - x, & x > 2. \end{cases}$

   (n) $\lim_{x\to -1^-} f(x)$ if $f(x) = \begin{cases} 1, & x \le -1 \\ x+2, & x > -1. \end{cases}$

   (o) $\lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} 3, & x \text{ an integer} \\ x+2, & \text{otherwise.} \end{cases}$

7. True or false? Prove your answers.

   (a) If $\lim_{x\to c}\big(f(x) + g(x)\big)$ exists but $\lim_{x\to c} f(x)$ does not exist, then $\lim_{x\to c} g(x)$ does not exist.

   (b) If $\lim_{x\to c}\big(f(x) + g(x)\big)$ and $\lim_{x\to c} f(x)$ exist, then $\lim_{x\to c} g(x)$ exists.

   (c) If $\lim_{x\to c} \sqrt{f(x)}$ exists, then $\lim_{x\to c} f(x)$ exists.

   (d) If $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} \sqrt{f(x)}$ exists.

   (e) If $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} \dfrac{1}{f(x)}$ exists.

   (f) If $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ exists and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} f(x) = 0$.