Calculus — Homework 3 (Fall 2025)

1. Let $(a_n)_{n=1}^{\infty}$ be a bounded increasing sequence, and $\alpha = \sup\{a_1, a_2, a_3, \cdots\}$. Prove that $\lim_{n\to\infty} a_n = \alpha$.

2. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence, and $A_n = \{a_n, a_{n+1}, \cdots\}$.
   (a) Prove that $\overline{\lim_{n\to\infty}}\, a_n = \inf\{\sup A_n \mid n \in \mathbb{N}\}$ and $\underline{\lim_{n\to\infty}}\, a_n = \sup\{\inf A_n \mid n \in \mathbb{N}\}$.
   (b) Let $\alpha = \overline{\lim_{n\to\infty}}\, a_n$. Prove that for each $\epsilon > 0$ there exists N_ϵ such that $a_n < \alpha + \epsilon$ for any $n \geq N_\epsilon$.
   (c) Prove that if $(a_{\hat{n}_m})_{m=1}^{\infty}$ be a convergent subsequence of $(a_n)_{n=1}^{\infty}$, then $\lim_{m\to\infty} a_{\hat{n}_m} \leq \alpha$.
   (d) Prove that for each $\epsilon > 0$ there exists n_ϵ such that $a_{n_\epsilon} > \alpha - \epsilon$.
   (e) Prove that there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ such that $\lim_{k\to\infty} a_{n_k} = \alpha$.
   (f) Prove that there exists a subsequence $(a_{\tilde{n}_l})_{l=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ such that $\lim_{l\to\infty} a_{\tilde{n}_l} = \underline{\lim_{n\to\infty}}\, a_n$.
   (g) Prove that if $(a_{\hat{n}_m})_{m=1}^{\infty}$ be a convergent subsequence of $(a_n)_{n=1}^{\infty}$, then $\underline{\lim_{n\to\infty}}\, a_n \leq \lim_{m\to\infty} a_{\hat{n}_m} \leq \overline{\lim_{n\to\infty}}\, a_n$.

3. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be bounded sequences.
   (a) Prove that $\overline{\lim_{n\to\infty}}(a_n + b_n) \leq \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n$.
   (b) Prove that there exist bounded sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $\overline{\lim_{n\to\infty}}(a_n + b_n) \neq \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n$.

4. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence, and $A_n = \{a_n, a_{n+1}, \cdots\}$.
   (a) Suppose that $(a_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} a_n = \alpha$. Prove that for each $\epsilon > 0$, there exists N_ϵ such that
   $$\alpha - \epsilon < \sup A_n < \alpha + \epsilon$$
   for any $n \geq N_\epsilon$.
   (b) Suppose that $(a_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} a_n = \alpha$. Prove that $\overline{\lim_{n\to\infty}}\, a_n = \underline{\lim_{n\to\infty}}\, a_n = \alpha$.
   (c) Suppose that $\overline{\lim_{n\to\infty}}\, a_n = \underline{\lim_{n\to\infty}}\, a_n = \alpha$. Prove that $(a_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} a_n = \alpha$.

5. Let a, b be real numbers such that $a < b$. Define a sequence $(x_n)_{n=0}^{\infty}$ by
   $$\begin{cases} x_{n+1} = \dfrac{x_n + x_{n-1}}{2}, & \forall n \geq 1, \\ x_0 = a, \quad x_1 = b. \end{cases}$$
   Prove that $(x_n)_{n=0}^{\infty}$ is convergent.

6. Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence with the properties:
   (i) There exist $M, N > 0$ such that $a_n \leq -M$ for all $n \geq N$.
   (ii) The sequence $(a_n^2)_{n=1}^{\infty}$ is a Cauchy sequence.

   Prove that the sequence $(a_n)_{n=1}^{\infty}$ is also a Cauchy sequence. (Hint: $a - b = \dfrac{a^2 - b^2}{a + b}$.)

7. Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to\infty} a_n = \alpha$. Define the average sequence by
   $$\sigma_n = \frac{a_1 + \cdots + a_n}{n}.$$
   (a) Prove that for any $\epsilon > 0$, there exists N_ϵ such that for all $n \geq N_\epsilon$, the following inequality holds:
   $$\frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n - N_\epsilon)(\alpha - \epsilon)}{n} < \sigma_n < \frac{a_1 + \cdots + a_{N_\epsilon}}{n} + \frac{(n - N_\epsilon)(\alpha + \epsilon)}{n}.$$
   (b) Prove that the sequence $(\sigma_n)_{n=1}^{\infty}$ is also convergent, and $\lim_{n\to\infty} \sigma_n = \alpha$.
   (c) Prove that there exists a divergent sequence $(a_n)_{n=1}^{\infty}$ whose associated average sequence $(\sigma_n)_{n=1}^{\infty}$ is convergent.