Calculus — Homework 2 (Fall 2025)

1. Prove that if $(a_n = c)_{n=1}^{\infty}$ is a constant sequence, then $\lim_{n\to\infty} a_n = c$.
2. Prove that
   (a) the sequence $\left(\frac{1}{\sqrt{n}}\right)_{n=1}^{\infty}$ converges;
   (b) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.
3. Prove that the sequence $\left(\frac{\sin(n)+\cos(n)}{\sqrt{n}}\right)_{n=1}^{\infty}$ converges.
4. Prove that the sequence $\left(\frac{n^4-3n^2+n+2}{n^3-7n}\right)_{n=1}^{\infty}$ diverges.
5. State whether the sequence converges and, if it does, find the limit.
   (a) $a_n = 2^n$.
   (b) $a_n = \frac{4^n}{\sqrt{n^2+1}}$.
   (c) $a_n = 2^{2/n}$.
   (d) $a_n = \left(\frac{2}{n}\right)^n$.
   (e) $a_n = \frac{4^{100n}}{n!}$.
   (f) $a_n = \left(\frac{1}{2}+\frac{3}{n}\right)^{3n}$.
6. Prove that a sequence converges to L if and only if every subsequence of it converges to L.
7. Prove that the sequence $\left(\cos\left(\frac{n\pi}{3}\right)\right)_{n=1}^{\infty}$ diverges.
8. Let
$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$
   (a) Prove that the sequence $(x_n)_{n=1}^{\infty}$ converges.
   (b) Let $e = \lim_{n\to\infty} x_n$. Prove that
$$0 < e - x_n < \frac{3}{(n+1)!}.$$
   (c) Prove that e is an irrational number. (Hint: $x_n = \frac{p_n}{n!}$ for some integer p_n.)
9. Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers. Define
$$s_n = a_1 + \cdots + a_n,$$
$$t_m = a_1 + 2a_2 + 4a_4 + \cdots + 2^m a_{2^m} = \sum_{k=0}^{m} 2^k \cdot a_{2^k}.$$
   (a) Prove that if $(s_n)_{n=1}^{\infty}$ is convergent, then so is $(t_m)_{m=1}^{\infty}$.
   (b) Prove that if $(t_m)_{m=1}^{\infty}$ is convergent, then so is $(s_n)_{n=1}^{\infty}$.